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D I F F E R E N T I A L E Q U A T I O N S
W I T H F I X E D C R I T I C A L P O I N T S

Part I : Second order equations

by

F. J. BUREAU

Professor of Mathematics
University of Liege, Belgium.

March 1963

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I. Introduction .

1. One of the main subjects of Analysis is the integration of differential equations , ordinary and partial . However, except for a few simple cases , the integration is very difficult .

The theory of analytic functions in one or more complex variables , initiated by Cauchy , Weierstrass and Riemann was applied by them to the study of differential equations.

Subsequent researches gave important results which will be described very briefly .

Consider a system of ordinary differential equations

$$(1) \quad \dot{y} = f(x,y) \quad ,$$

where y and f are vectors in an n -dimensional space . Suppose that the components of f are holomorphic in a neighborhood of $x = x_0$, $y = y_0$; then , the differential system (1) has one and only one solution $y(x) = y(x; y_0, x_0)$ such that $y(x_0) = y_0$. This function $y(x)$ may be continued analytically and gives rise to an analytic function of x , the singular points of which have to be determined .

When (1) is linear in y , the results are as follows :
i. the singular points of $y(x)$ are among the singular points of the coefficients of the system ; thus they are fixed and

determined directly by the equations ;

ii. the set of all solutions of (1) is a linear vector space so that the dependence of $y(x)$ on the initial values is known.

Suppose that $x = x_0$ is a regular singular point in the sense of L. Fuchs . Then the analytical character of the integral in the neighborhood of $x = x_0$ is completely determined by the indicial equation and accordingly is given by a set of constants which are easily obtained from the given system.

2. More complicated circumstances arise when (1) is not linear in y . In general , it is not possible to recognize if a given value x_0 of x is regular or singular for $y(x)$; the analytical character of $y(x)$ at x_0 may depend on the value y_0 of $y(x_0)$. Moreover , the singular points of $y(x)$ may be parametric (or movable) i.e. may depend on y_0 ; they may also be algebraic or transcendental , or even essential points and may be isolated or not .

Therefore , the following problem initiated by L. Fuchs as a consequence of his researches on linear differential equations is of importance : to determine all the equations

$$(2) \quad R(\dot{y}, y, x) = 0 ,$$

where R is a polynomial in \dot{y} and y with analytic coefficients in x , whose integral has no parametric critical points . In the neighborhood of every movable singularity , $y(x)$ must be

one-valued . To abbreviate , we shall say that an equation of this type is stable together with its integrals .

This problem was considered by Abel and Jacobi when

$$R \equiv \dot{y}^2 - (1 - y^2)(1 - k^2 y^2) ,$$

(k a constant) and by Briot et Bouquet when R is independent of x .

The solution of this problem was given by L. Fuchs , Poincaré and Painlevé and is as follows : the integral of the stable equations (2) are determined algebraically , or by quadratures , or depend on a Riccati equation . Consequently, they are reducible to classical transcendents , i.e. to algebraic and elliptic functions or to functions defined by linear differential equations .

3. To define new transcendental functions , one must therefore consider differential equations of higher order . However , new complications appear , of which the most serious is the possible existence of parametric transcendental or even essential singular points . For instance , in the latter case , $y(x)$ comes arbitrarily close to any complex value in every neighborhood of x_0 so that $\lim_{x \rightarrow x_0} y(x)$ is not uniquely determined .

Few and inconclusive results were obtained by Picard and Mittag-Leffler concerning stable equations of the second order.

Painlevé was the first to attack successfully this problem and to overcome the difficulties . To this effect , Painlevé has

- i. to obtain a set of necessary conditions for the absence of parametric critical points ;
- ii. to show that the necessary conditions thus obtained are or are not sufficient .

Consider for instance the equation

$$(3) \quad \ddot{y} = R(\dot{y}, y, x)$$

where R is a rational function of \dot{y} and y , with analytic coefficients in x . In order to obtain a set of necessary conditions for stability , Painlevé introduces in (3) by a suitable transformation of y and x , a parameter ε in such a way that the new equation has the same fixed critical points as (3) and is integrable when $\varepsilon = 0$. Then $y(x, \varepsilon)$ may be developed into a series of ascending powers of ε ,

$$y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots ,$$

the coefficients of which are also uniform and determined by quadratures ; the conditions that these functions be stable are necessary conditions for the stability of (3) .

This method was used by Painlevé and Gambier to determine all the stable equations (2) . The result is as follows :

The stable equations (2) are integrable in terms of classical transcendents or may be reduced by a transformation

$$(4) \quad u = \frac{ay + b}{cy + d}, \quad z = \varphi(x),$$

where a, b, c, d, φ are analytic functions of x , to one of the six canonical equations in the following table :

Table I

$$I \ 1. \quad \ddot{y} = 6 y^2 + x,$$

$$I \ 2. \quad \ddot{y} = 2 y^3 + x y + \alpha,$$

$$I \ 3. \quad \ddot{y} = \frac{\dot{y}^2}{y} - \frac{\dot{y}}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y},$$

$$I \ 4. \quad \ddot{y} = \frac{\dot{y}^2}{2y} + \frac{3}{2} y^2 + 4 x y^2 + 2(x^2 - \alpha)y + \frac{\beta}{y},$$

$$I \ 5. \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 - \frac{\dot{y}}{x} + \frac{(y-1)^2}{x^2} \left(\alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{x} + \delta \frac{y(y+1)}{y-1},$$

$$I \ 6. \quad \ddot{y} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \dot{y}^2 - \left(\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x-y} \right) \dot{y} \\ + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[\alpha + \frac{\beta x}{y^2} + \frac{\gamma (x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2} \right];$$

$\alpha, \beta, \gamma, \delta$ are constants .

As was shown by Painlevé , the integrals of the equations of Table I are not reducible to classical transcendents (except possibly for particular values of $\alpha, \beta, \gamma, \delta$) and thus define new transcendental functions .

4. The method used by Painlevé , although theoretically simple , implies intricate calculations . Because of these difficulties and because of the great number of cases that have to be considered , Painlevé found only three equations of Table I (namely I 1,2,3) ; the three other equations were discovered by Gambier .

The method of Painlevé may also be applied to equations of order higher than two ; however , the intricacies increase with the order of the equation .

To reduce these difficulties to a minimum , another method will be developed in this paper ; it will systematically avoid, as far as possible , any integration of systems of linear differential equations and will reduce the problem to elementary algebraic processes . This method was initiated by the author in 1939 ; a few examples were given at that time .

The ultimate purpose of the actual theory is : to find out if a given equation of the form (3) is stable and if so , to integrate this equation .

To solve this problem , one may write all the equations (3) obtained from the canonical equations by a transformation (4) . However , this new Table of equations is far too extended to be written in full . According to Gambier , certain sub-classes give rise to more than one hundred equations .

Fortunately , our method associates with each given

equation , a small number of integers , easily determined and giving immediately the corresponding sub-class ; to perform the integration , one has only to refer to the appropriate paragraph of the paper . The situation is thus similar to the theorem of Fuchs in case of regular integrals of a linear differential equation .

Finally , we observe that our method applies also to equations of order higher than two , not necessarily linear in the derivative of higher order . This will be the subject of another paper .

I. Statement of the problem .

1. Let x be a complex independent variable and ε a complex parameter . Let $y = (y_1, \dots, y_n)$ be a set of functions satisfying the system of ordinary differential equations

$$(1.1) \quad \frac{dy}{dx} = A(x; y; \varepsilon)$$

where $A = (A_1, \dots, A_n)$ is a given vector-function of x, y, ε , holomorphic within a certain domain D containing the point $x = x_0, y = y^0 = (y_1^0, \dots, y_n^0), \varepsilon = 0$.

When $\varepsilon = 0$, the system

$$(1.2) \quad \frac{dy}{dx} = A(x, y; 0)$$

is called the reduced (or non perturbed , or undisturbed) system corresponding to the perturbed system (1.1) .

It is known that the system of differential equations (1.1) admits a unique solution $y(x) = y(x; \varepsilon) = y(x - x_0; y_0; \varepsilon)$ which is holomorphic within a certain domain containing the point $x_0, y_0, \varepsilon = 0$ and which reduces to y_0 when $x = x_0$.

On using a theorem of Poincaré , basic in perturbation theory , one may write

$$(1.3) \quad y(x) = y(x; \varepsilon) = v(x) + \sum_{p=1}^{\infty} u_p(x) \varepsilon^p$$

where $v(x) = (v_1, \dots, v_n)$ is a solution of the reduced equation

$$(1.4) \quad \frac{dv}{dx} = A(x;v;0) .$$

This solution $v(x)$ depends on n arbitrary constants c_1, \dots, c_n (for instance y_1^0, \dots, y_n^0) and is holomorphic for certain values of these constants c_k when x varies along a curve L joining two points of the domain D .

On setting the matrix

$$(1.5) \quad B(x;v) = \frac{\partial A_i(x;v;0)}{\partial v_k} ,$$

one may write

$$(1.6) \quad \begin{aligned} A(x;v + \sum_{p=1}^{\infty} \epsilon^p u_p; \epsilon) &= A(x;v;0) + \epsilon [B(x;v)u_1 + C_1(x;v)] \\ &+ \sum_{k=2}^{\infty} \frac{\epsilon^k}{k!} [B(x;v) u_k + C_k(x;v, u_1, \dots, u_{k-1})] , \end{aligned}$$

where $C_k(x;v, u_1, \dots, u_{k-1})$ is a vector with components depending only on the components of the vectors v, u_1, \dots, u_{k-1} .

Therefore, u_k is a solution of

$$(1.7) \quad \frac{du_k}{dx} = B(x;v) u_k + C_k(x;v, u_1, \dots, u_{k-1}) ,$$

$$(k=1, 2, \dots) .$$

The solution of this non homogeneous system of linear differential equations may be obtained by the method of variation of parameters when the solution of the homogeneous system

$$(1.8) \quad \frac{du}{dx} = B(x;v) u$$

is known .

Moreover , if $v(x;c)$, $(c=c_1, \dots, c_n)$, is the general solution of (1.4) , depending on the arbitrary constants c_1, \dots, c_n , one has

$$(1.9) \quad \frac{d}{dx} \frac{\partial v}{\partial c_k} = B(x;v) \frac{\partial v}{\partial c_k} , \quad (k=1, \dots, n)$$

so that $\frac{\partial v}{\partial c_1} , \dots , \frac{\partial v}{\partial c_n}$ are n linearly independent solu-

tions of (1.8) .

2. Classification of singularities . The solution $y(x) \equiv y(x-x_0, y_0; \varepsilon)$ of (1.1) together with the series obtained by analytical continuation defines a function - call it again $y(x)$ - which is a solution of (1.1) and in which x_0 , y_0 appear as parameters . This function $y(x)$ may have singular points depending or not on the initial value y_0 of y . Those singular points of $y(x)$ independent of y_0 are called fixed or intrinsic singular points of $y(x)$ or of the differential equations (1.1) ; those singular points of $y(x)$ which depend on y_0 are known as the movable or parametric singular points of $y(x)$ or of the differential equations (1.1).

When the integral $y(x)$ has no parametric critical points (i.e. branch points or essential singularities) , then we

say that $y(x)$ is a stable integral of (1.1) ; when the differential system (1.1) has only stable integrals , we say that this system is stable . In the other cases , the differential system and the corresponding integrals are unstable .

The following problem then arises : to find necessary and sufficient conditions in order that a given system (1.1) be stable .

To solve this problem , the following theorem is essential:

General theorem of stability : If the general solution of the differential system (1.1) is uniform in x for all values of ε in D except possibly $\varepsilon = 0$, then it will also be uniform for $\varepsilon = 0$. Moreover , the coefficients $u_p(x), (p=1,2,\dots)$, of the series (1.3) are also uniform .

To prove this theorem , consider in the x -plane , a closed path L beginning and ending at x_0 , on which $y(x,\varepsilon)$ is analytic . Let $u_k(x)$ be the first of the functions $v(x), u_p(x)$, $(p=1,2,\dots)$, which takes on two (or more) values at x_0 ; then , one has

$$y(x,\varepsilon) = v(x) + \varepsilon u_1 + \dots + \varepsilon^{p-1} u_{p-1} + \varepsilon^p \left[u_p + \varepsilon u_{p+1} + \dots \right]$$

$$\equiv Y_1(x,\varepsilon) + \varepsilon^p Y_2(x,\varepsilon) .$$

When ε is small enough , the values of $Y_2(x,\varepsilon)$ are very near to the values of $u_p(x)$; therefore , if $u_p(x)$ is multi-valued at x_0 , then $Y_2(x,\varepsilon)$ is also multi-valued at x_0 . Furthermore, the number of values of $y(x,\varepsilon)$ at x_0 is not less than the

numbers of values of any of the functions $v(x), u_k(x), (k=1,2,\dots)$, at x_0 . Consequently, if $y(x,\epsilon)$ is uniform, then the functions $v(x), u_k(x), (k=1,2,\dots)$ are also uniform.

The method initiated by P. Painlevé to solve the problem breaks up into two parts. First, a set of necessary conditions for the absence of parametric critical points is obtained and a set of equations satisfying these necessary conditions is derived; second, it is shown by direct integration or by using an appropriate method that these particular equations are stable.

To obtain the necessary conditions, a parameter ϵ is introduced into the differential system under consideration, in such a way that the new system has only fixed critical points and is integrable when $\epsilon = 0$. The functions $v(x), u_k(x)$ are solutions of systems of linear differential equations and thus determined by quadratures. The conditions that their parametric branch points are fixed give rise to necessary conditions for the stability of the given system.

The necessary conditions thus obtained enable one to simplify the given system; the same method is again applied until no further necessary conditions of stability are obtained.

This method, theoretically simple, requires heavy and tedious calculations.

To avoid these cumbersome calculations, another method is developed in what follows; it systematically avoids, as

far as possible , any integration of systems of linear differential equations and reduces the problem to elementary algebraic processes .

II. The theorems of stability .

3. In the next paragraphs , we shall apply the general theorem of stability to various important particular cases ; the results thus obtained will prove very useful in what follows .

Throughout this paper , we denote by dots differentiations with respect to the independent variable x .

Let $k \gg 0$ be an integer and $A \neq 0$ a constant .

Theorem I . In order that the equation

$$(3.1) \quad y^k \dot{y} = A$$

be stable , it is necessary and sufficient that $k = 0$.

Indeed , the general solution of (3.1) is given by

$$(3.2) \quad y^{k+1} = y_0^{k+1} + A(x - x_0)$$

where x_0 and y_0 are arbitrary constants ; it is clear that $y(x)$ defined by (3.2) is stable if and only if $k=0$.

Theorem II. In order that the equation

$$(3.3) \quad \dot{y} = A y^k$$

be stable , it is necessary and sufficient that $k \leq 2$.

Indeed , the general solution of (3.2) is given by

$$(3.4) \quad y^{1-k} = (1-k) [A(x - x_0) + c] ,$$

where x_0 and c are arbitrary constants . In order that $y(x)$ defined by (3.4) be stable , it is necessary and sufficient that $1 - k = \frac{1}{n}$ where n is an integer ; then $k \leq 2$.

Theorem III. Let $P(y)$ be a holomorphic function of y in a neighborhood of $y = 0$. In order that the system of differential equations

$$(3.5) \quad \begin{aligned} \dot{y} &= z , \\ y^k \dot{z} &= A z^2 [1 + P(y)] \end{aligned}$$

be stable , it is necessary

- i. that $k = 0$ or 1 ;
- ii. and if $k = 1$, that $A = 1 - \frac{1}{n}$, where $n \neq 0$ is a positive or negative integer or $n = \infty$ (i.e. $A = 1$) .

Indeed , set

$$\begin{aligned} y &= \varepsilon u = \varepsilon u_1 + \varepsilon^k u_2 + \dots , \\ z &= \varepsilon^k v = \varepsilon^k v_1 + \varepsilon^{k+1} v_2 + \dots ; \end{aligned}$$

the differential system (3.5) becomes

$$\begin{aligned} \dot{u} &= \varepsilon^{k-1} v , \\ u^k \dot{v} &= A v^2 [1 + P(\varepsilon u)] . \end{aligned}$$

It then follows that u_1, u_2, v_1, \dots satisfy the following equations

$$\dot{u}_1 = 0 , \quad u_1^k \dot{v}_1 = A v_1^2 , \quad \dot{u}_2 = v_1 ;$$

therefore , if a, b, c are arbitrary constants with the restriction that $b, c \neq 0$, one has

$$u_1(x) = b \quad , \quad v_1(x) = \frac{b^k c}{b^k - A c(x-a)} \quad ,$$

$$u_2(x) = \int_a^x v_1(t) dt = \frac{b^k}{A c} \lg \frac{b^k}{b^k - A c(x-a)} \quad .$$

In order that $u_2(x)$ be stable , it is necessary that $k = 0$ or 1 .

Now , suppose $k = 1$ and set $x = a + \varepsilon t$, $y = \varepsilon u$, $z = z$, where a is a constant . Substitution in (3.5) gives

$$\frac{du}{dt} = z \quad , \quad u \frac{dz}{dt} = A z^2 [1 + P(\varepsilon u)] \quad .$$

When $\varepsilon = 0$, one finds the reduced system

$$(3.6) \quad \frac{du}{dt} = z \quad , \quad u \frac{dz}{dt} = A z^2 \quad .$$

The function $u(t)$ defined by (3.6) satisfies the equation

$$u \frac{d^2 u}{dt^2} = A \left(\frac{du}{dt} \right)^2$$

and implies :

$$i. \text{ if } A = 1 \quad , \quad u = e^{ct} + c_1 \quad ;$$

$$ii. \text{ if } A \neq 1 \quad , \quad u = (c_1 + ct)^{1/(1-A)} \quad ;$$

c and c_1 are arbitrary constants . Therefore , in order that $u(t)$ be stable , it is necessary that $A = 1$ or $\frac{1}{1-A} = n$, where $n \neq 0$ is a positive or negative integer [note that $n = \infty$ gives $A = 1$] .

5. In this and in the next paragraphs , let p denote a constant , $k \geq 0$ an integer , $P(x;z)$ a polynomial in z of degree $\leq k-2$, and $H_1(x;z;u)$, $(i=1,2,\dots)$, a polynomial in u . Further , we suppose that $h(x;z)$, $H_1(x;z;u)$ are holomorphic functions of z at $z = 0$ and that $P(x;z)$, $h(x;z)$, $H_1(x;z;u)$ are analytic functions of x in a given domain D .

We consider the differential system

$$(5.1) \quad \begin{cases} \dot{z} = 1 + z P(x;z) + z^k u \\ z \dot{u} = p u + h(x;z) + z H_1(x;z;u) \end{cases} ,$$

concerning which we have the following basic theorem :

Theorem IV . In order that the differential system (5.1) be stable , it is necessary

- i. that p be an integer , positive , negative , or zero ,
- ii. and if $p = 0$, that $h(x;0) \equiv 0$.

Proof . Set $x = a + \varepsilon t$, $z = \varepsilon v$; $a \in D$ is a constant . Substitution in (5.1) yields

$$\begin{aligned} \frac{dv}{dt} &= 1 + \varepsilon v P(a+\varepsilon t, \varepsilon v) + \varepsilon^k v u , \\ v \frac{du}{dt} &= p u + h(a+\varepsilon t; \varepsilon v) + \varepsilon v H_1(a+\varepsilon t, \varepsilon v; u) . \end{aligned}$$

When $\varepsilon = 0$, one finds the reduced system

$$(5.2) \quad \frac{dv}{dt} = 1 , \quad v \frac{du}{dt} = p u + h(a;0) .$$

Suppose $p \neq 0$. Then the general solution of (5.2) is

$$v(t) = t - b , \quad u(t) = c(t-b)^p - \frac{h(a;0)}{p} ;$$

b and c are arbitrary constants .

In order that $u(t)$ be stable , it is necessary that p be an integer , positive , negative , or zero .

Now suppose $p = 0$; the non perturbed system (5.2) becomes

$$\frac{dv}{dt} = 1 \quad , \quad v \frac{du}{dt} = h(a;0)$$

whose general solution is

$$v(t) = t - b \quad , \quad u(t) = c + h(a;0) \lg(t-b) \quad ,$$

where b and c are again arbitrary constants . Therefore , in order that $u(t)$ be stable , it is necessary that $h(a;0) = 0$; because $a \in D$ is arbitrary , this condition becomes $h(x;0) = 0$.

6. In what follows , we shall apply our basic theorem IV to the differential system

$$(6.1) \quad \begin{cases} \dot{z} = 1 + z u \quad , \\ z \dot{u} = p u + h(x;z) + z H_1(x;z;u) \quad , \end{cases}$$

where p is an integer and $h(x;z)$, $H_1(x,z;u)$ have the properties indicated above [see § 5] .

When p is a negative integer , no condition for stability follows from theorem IV .

Suppose $p > 0$ and set

$$(6.2) \quad u = P(x;z) + z^p v \quad ,$$

where v is an unknown function and $P(x;z)$ is a polynomial in z of degree $p-1$ whose coefficients will be determined later on. Substituting in (17) , one finds

$$\dot{z} = 1 + z P(x; z) + z^{p+1} v$$

so that , from (6.2) , it follows that

$$\dot{u} = Q(x; z) + z^p \dot{v} + p z^{p-1} v + z^p H_2(x, z; v) ;$$

$Q(x; z)$ is a polynomial in z of degree $\leq p+1$; $H_2(x; z; v)$ has the same properties as $H_1(x, z; u)$.

Now , one may write

$$\begin{aligned} H_1(x, z; u) &= H_1(x, z; P + z^p v) \\ &= R(x, z) + z^p H_3(x, z; v) ; \end{aligned}$$

$R(x, z)$ is a polynomial in z of degree $\leq p-1$ and is an analytic function of x ; $H_3(x, z; v)$ has the same properties as $H_1(x, z; u)$. Then , the second equation (6.1) becomes

$$z^{p+1} \dot{v} = E(x; z) + z^{p+1} H_4(x, z; v) ,$$

where $H_4 = H_3 - H_2$ and

$$E(x; z) = p P(x; z) - z Q(x; z) + h(x; z) + z R(x; z)$$

is a polynomial in z of degree p .

Suppose that we determine the p coefficients of the polynomial $P(x; z)$ in order that $E(x; z) \equiv \ell(x) z^p$; then , the differential system (6.1) becomes

$$(6.3) \quad \begin{cases} \dot{z} = 1 + z P(x; z) + z^{p+1} v , \\ z \dot{v} = \ell(x) + z H_4(x, z; v) . \end{cases}$$

Then the condition for stability , $\ell(x) \equiv 0$, follows from theorem IV .

It is clear that when $\ell(x) \equiv 0$, the system (6.3) has a unique holomorphic solution $z(x)$, $v(x)$ such that $z(x_0) = 0$, $v(x_0) = v_0$, where v_0 is an arbitrary constant .

Note that the condition $\mathcal{L}(x) \equiv 0$ is also sufficient in order that the general solution of the differential system (6.3) have no branch point in x_0 . Therefore, no additional condition for stability can be obtained by the preceding method; its efficacy is exhausted.

7. It is particularly easy to dispose of the above method when p is small. For future use, it will be useful to consider the values $p = 0, 1, 2, 3$.

Suppose that the second equation of (6.1) is written more explicitly in the form

$$(7.1) \quad \begin{aligned} z \dot{u} = & p u + A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \dots \\ & + u(B_1 z + B_2 z^2 + \dots) + k z u^2, \end{aligned}$$

where the A_i 's and B_i 's are analytic functions of x and k is a number independent of x .

i. $p = 0$. The condition for stability is $A_0 = 0$.

ii. $p = 1$. Set $u = \alpha + z v$, where α is to be determined later on. Substituting in (7.1), one obtains the two relations

$$(7.2) \quad \begin{cases} \alpha + A_0 = 0 \\ \dot{\alpha} = A_1 + \alpha B_1 + k \alpha^2. \end{cases}$$

On eliminating α , it follows that the condition for stability is

$$(7.3) \quad \dot{A}_0 + A_1 - A_0 B_1 + k A_0^2 = 0.$$

iii. $p = 2$. Set $u = \alpha + \beta z + z^2 v$, where α and β are to

be determined later on . Substituting in (7.1) , one obtains the three relations

$$(7.4) \quad \begin{cases} 2\alpha + A_0 = 0 , \\ \dot{\alpha} = \beta + A_1 + \alpha B_1 + k \alpha^2 , \\ \dot{\beta} = A_2 + \alpha B_2 + \beta B_1 + (2k-1)\alpha\beta . \end{cases}$$

To obtain the condition for stability , one must eliminate α and β from this set of equations .

iv. $p = 3$. Set $u = \alpha + \beta z + \gamma z^2 + z^3 v$; the same method gives rise to the four relations

$$(7.4) \quad \begin{cases} 3\alpha + A_0 = 0 , \\ \dot{\alpha} = 2\beta + A_1 + \alpha B_1 + k \alpha^2 , \\ \dot{\beta} = \gamma + A_2 + \alpha B_2 + \beta B_1 + (2k-1)\alpha\beta , \\ \dot{\gamma} = A_3 + \alpha B_3 + \beta B_2 + \gamma B_1 + (k-1)(\beta^2 + 2\alpha\gamma) , \end{cases}$$

from which the condition for stability follows by the elimination of α , β , γ .

In general , the process of elimination is complicated . Fortunately when applied to particular equations , the condition for stability is , in most cases , readily obtained and easy to work with . Examples will be given later on .

III. Applications .

8. To find all the stable equations of the form

$$(8.1) \quad \ddot{y} = \frac{P(y)}{Q(y)} \dot{y}^2 ,$$

where $P(y)$, $Q(y)$ are polynomials in y independent of x .

1. $Q(y)$ is a constant or has only simple roots.

Let $y = a$ be a root of $Q(y)$ of multiplicity $k \gg 1$. The equation (8.1) is equivalent to the differential system

$$(8.2) \quad \dot{y} = z, \quad \dot{z} = \frac{A z^2}{(y - a)^k} [1 + g(y)]$$

where A is a constant and $g(y)$ is a holomorphic function of y in a neighborhood of $y = a$ and such that $g(a) = 0$.

According to theorem III, a necessary condition for this system (26) to be stable is $k = 1$ and $A = 1$ or $1 - \frac{1}{n}$, where $n \neq 0$ is a positive or negative integer.

ii. Let a_1, \dots, a_p be the p simple roots of $Q(y)$. Denote by A_k the residue of $\frac{P(y)}{Q(y)}$ at $y = a_k$; one has $A_k = 1$ or

$A_k = 1 - \frac{1}{n_k}$, where $n_k \neq 0$ is a positive or negative integer

(cf. i). Then (8.1) may be rewritten as

$$(8.3) \quad \ddot{y} = \dot{y}^2 \left[\sum_{k=1}^p \frac{A_k}{y - a_k} + R(y) \right],$$

where $R(y)$ is a polynomial in y .

Now, set $y = u^{-1}$; equation (8.3) becomes

$$\ddot{u} = \frac{\dot{u}^2}{u} \left[2 - \sum_{k=1}^p \frac{A_k}{1 - a_k u} + \frac{1}{u} R\left(\frac{1}{u}\right) \right] \equiv \dot{u}^2 B(u).$$

According to theorem III, $u = 0$ must be a simple pole of $B(u)$ so that $\frac{1}{u} R\left(\frac{1}{u}\right) = b + O(u)$, where b is a constant.

Therefore , the polynomial $R(y) = \frac{b}{y} + C \left(\frac{1}{y} \right)$ is identically zero .

Moreover , it follows from theorem III that

$$A = \sum_{k=1}^p A_k$$

is equal to 1 or to $1 - \frac{1}{n}$, where $n \neq 0$ is a positive or negative integer .

9. It follows from the preceding paragraph , that to obtain all the stable equations of the form (8.1) , one has to find the set of integers n , n_k , ($k=1, \dots, p$) different from zero and satisfying the equation

$$(9.1) \quad \sum_{k=1}^p \left(1 - \frac{1}{n_k} \right) = 1 - \frac{1}{n} .$$

Note that n and n_k may be infinite .

i. Suppose $A = 1$ or $n = \infty$; then

$$\sum_{k=1}^p \left(1 - \frac{1}{n_k} \right) = 1$$

or

$$p - 1 = \frac{1}{n_1} + \dots + \frac{1}{n_p} \leq \frac{p}{2} ;$$

therefore $p \leq 2$ and

a. $p = 1$, $n_1 = \infty$,

b. $p = 2$, $n_1 = n_2 = 2$.

ii. Suppose $A > 1$; then $A = 1 - \frac{1}{n} \leq 2$. Because

$$A_k = 1 - \frac{1}{n_k} \gg \frac{1}{2}, \text{ one has } p \leq 4.$$

a. $p = 4$; then $A = 2$, $A_k = \frac{1}{2}$, $(k=1,2,3,4)$, i.e.

$$n = -1, \quad n_1 = n_2 = n_3 = n_4 = 2.$$

b. $p = 0$; then $A_k = 0$, $(k=1,2,3,4)$.

c. $p = 1$; then $A = A_1 = 1 - \frac{1}{n}$, n is an integer $\neq 0, 1$.

d. $p = 2$; then $\frac{1}{n} = \frac{1}{n_1} + \frac{1}{n_2} - 1 \leq 0$ so that n is negative.

On setting $n = -m$, $m > 0$, one has

$$(9.2) \quad \frac{1}{m} + \frac{1}{n_1} + \frac{1}{n_2} = 1$$

whose solutions are

$$(3,3,3) \quad , \quad (2,4,4) \quad , \quad (2,3,6) \quad , \quad (2,2,\infty) \quad , \quad (1,n,-n),$$

where n is an integer .

e. $p = 3$; then $\frac{1}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - 2 \leq \frac{3}{2} - 2 < 0$

and n is negative . On setting $n = -m$, $m > 0$, one has

$$(9.3) \quad \frac{1}{m} = 2 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \gg \frac{1}{2}$$

[note that $n_k \neq 1$] ; therefore $m = 1$ or 2 .

If $m = 1$, equation (9.3) reduces to (9.2) with $n = n_3$.

If $m = 2$, one has $n_1 = n_2 = n_3 = 2$. Note that $n_1 > 2$ is impossible because in that case $\frac{1}{n_1} < \frac{1}{2}$ implies

$$\frac{1}{n_2} + \frac{1}{n_3} > 1, \text{ an impossibility since } n_2, n_3 \text{ are different}$$

from 1 . Thus the solution is unique .

Corresponding to the solutions of equation (9.1) , there are nine possible types of equations of the form (8.1) . For convenience , we set

$$A = \frac{a}{ay + b} , \quad C = \frac{c}{cy + d} , \quad E = \frac{e}{ey + f} , \quad G = \frac{g}{gy + h}$$

where a, b, c, d, e, f, g, h are constants and eventually zero .

The nine possible values of $\frac{P(y)}{Q(y)}$ are tabulated below .

Table II .

$$\begin{aligned} 0 & ; \quad A & ; \quad A + C & ; \\ A(1 + \frac{1}{n}) + C(1 - \frac{1}{n}) & , \quad n > 1 \text{ an integer} & ; \\ \frac{1}{2} (A + C) + E & ; \quad \frac{2}{3} (A + C + E) & ; \\ \frac{1}{2} A + \frac{3}{4} (C + E) & ; \quad \frac{5}{6} A + \frac{2}{3} C + \frac{1}{2} E & ; \\ \frac{1}{2} (A + C + E + G) & . \end{aligned}$$

The corresponding differential equations are shown to be stable by direct integration . Indeed , they are equivalent to

$$\begin{aligned} \dot{y} &= 0 , \\ \dot{y} &= K(ay + b) , \\ \dot{y} &= K(ay + b)(cy + d) , \\ \dot{y}^n &= K(ay + b)^{n+1}(cy + d)^{n-1} , \\ \dot{y}^2 &= K(ay + b)(cy + d)(ey + f)^2 , \end{aligned}$$

$$\begin{aligned}
\dot{y}^3 &= K(ay + b)^2(cy + d)^2(ey + f)^2, \\
\dot{y}^4 &= K(ay + b)^2(cy + d)^3(ey + f)^3, \\
\dot{y}^6 &= K(ay + b)^5(cy + d)^4(ey + f)^3, \\
\dot{y}^2 &= K(ay + b)(cy + d)(ey + f)(gy + h),
\end{aligned}$$

where K is an arbitrary constant .

The solutions of these equations are known and involve only elementary or elliptic functions . Therefore, they are all stable ; the given list is exhaustive .

These results will be used later on .

10. For future use , we need the following theorem .

Denote by a and b two constants , not both zero , and by k and n two integers . Suppose $k \gg 1$; let $n \neq 0$ be positive or negative and eventually infinity .

Theorem V . In order that the equation

$$(10.1) \quad \ddot{y} = \left(1 - \frac{1}{n} \right) \frac{\dot{y}^2}{y} + a \frac{\dot{y}}{y^k} + \frac{b}{y^{2k-1}}$$

be stable , it is necessary that $k = 1$ and $n \neq 1$.

Proof . Note that equation (10.1) is invariant under the transformation $y = \varepsilon z$, $x = \varepsilon^k t$, where ε is a parameter ; note also that under this transformation , one has

$$y^{1-k} \frac{dx}{dy} = z^{1-k} \frac{dt}{dz} .$$

Accordingly , we set

$$u = y^{1-k} \frac{dx}{dy} ;$$

then equation (10.1) is equivalent to the system

$$y^{k-1} \frac{dy}{dx} = \frac{1}{u} , \quad y^k \frac{du}{dx} = P(u) ,$$

where

$$P(u) = \frac{1}{n} - k - a u - b u^2.$$

This system is also equivalent to the system

$$(10.2) \quad y^{k-1} \frac{dy}{dx} = \frac{1}{u} ,$$

$$(10.3) \quad y \frac{du}{dy} = u P(u) .$$

Suppose $k > 1$; then , the equation (10.3) has a solution $u = h$, where $h \neq 0$ is such that $P(h) = 0$. Then , from (10.2), one has $dx = h y^{k-1} dy$ and

$$y^k = \frac{k}{h} (x + c) ,$$

where c is an arbitrary constant ; this function $y(x)$ is not uniform .

Suppose $k = n = 1$; then , equation (10.1) is equivalent to the differential system

$$(10.4) \quad \dot{y} = \frac{1}{u} , \quad y \dot{u} + a u + b u^2 = 0 .$$

Set $x = \varepsilon t$, $u = \varepsilon v$ where ε is a parameter ; then , the differential system (10.4) may be rewritten as

$$\frac{dy}{dt} = \frac{1}{v} , \quad y \frac{dv}{dt} + \varepsilon v(a + b \varepsilon v) = 0 .$$

We may now apply the general theorem of stability . To do this , $y(t)$, $v(t)$ are developed as series of ascending powers of the parameter ϵ . If x_0 , y_0 , v_0 are arbitrary constants , one finds

$$y = y_0 + \frac{x - x_0}{v_0} + O(\epsilon) ,$$

$$v = v_0 - \epsilon a v_0^2 \lg \left(y_0 + \frac{x - x_0}{v_0} \right) + O(\epsilon^2) \quad \text{if } a \neq 0 ,$$

$$v = v_0 - \epsilon^2 b v_0^3 \lg \left(y_0 + \frac{x - x_0}{v_0} \right) + O(\epsilon^3) \quad \text{if } a = 0 .$$

It then follows readily that the differential system (10.4) and the equation (10.1) are not stable . Our theorem is proved .

IV. The equation $\ddot{y} = R(x, y, \dot{y})$.

11. Consider the equation

$$(11.1) \quad \ddot{y} = R(x, y, \dot{y}) ,$$

where $R(x, y, \dot{y})$ is a rational and irreducible function of y and \dot{y} , with coefficients analytic in x . Our purpose is to find all the stable equations of the type (11.1) .

The equation (11.1) is equivalent to the differential system

$$(11.2) \quad \dot{y} = z , \quad \dot{z} = R(x, y, z) .$$

Let x_0 , y_0 be particular (arbitrary) values of x , y ;

let $z = a(x_0, y_0)$ be a pole of order $m > 0$ of $R(x_0, y_0, z)$.

Set $z = u + a(x, y)$; then, in a neighborhood of x_0, y_0 , the differential system (11.2) may be rewritten as

$$(11.3) \quad \begin{cases} \dot{y} = u + a(x, y) \\ \dot{u} = u^{-m} P(x, y, u) \end{cases},$$

where $P(x, y, u)$ is a holomorphic function of x, y, u such that $P(x_0, y_0, 0) \neq 0$.

Let ε be a parameter. On setting

$$(11.4) \quad x = x_0 + \varepsilon^p t, \quad y = y_0 + \varepsilon^q w, \quad u = \varepsilon^r v,$$

the differential system (11.3) becomes

$$(11.5) \quad \begin{cases} \varepsilon^{q-p} \frac{dw}{dt} = \varepsilon^r v + a(x_0 + \varepsilon^p t, y_0 + \varepsilon^q w) \\ \varepsilon^{r(m+1)-p} v^m \frac{dv}{dt} = P(x_0 + \varepsilon^p t, y_0 + \varepsilon^q w, \varepsilon^r v) \end{cases},$$

so that, assuming $p = q$, $p = r(m+1)$, $r = 1$, one obtains

$$(11.6) \quad \begin{cases} \frac{dw}{dt} = a(x_0 + \varepsilon^p t, y_0 + \varepsilon^p w) + \varepsilon v \\ v^m \frac{dv}{dt} = P(x_0 + \varepsilon^p t, y_0 + \varepsilon^p w, \varepsilon v) \end{cases}.$$

When $\varepsilon = 0$, one finds the reduced system

$$\frac{dw}{dt} = a(x_0, y_0), \quad v^m \frac{dv}{dt} = P(x_0, y_0, 0);$$

for this system to be stable, it is necessary, according to theorem I, that $m = 0$.

Therefore, $R(x, y, \dot{y})$ is a polynomial in \dot{y} of degree s ;

let us write

$$R(x, y, \dot{y}) = R_0(x, y) + R_1(x, y) \dot{y} + \dots + R_s(x, y) \dot{y}^s .$$

To determine the value of the positive integer s , observe that the equation (11.1) is now equivalent to the differential system

$$(11.7) \quad \begin{cases} \dot{y} = z , \\ \dot{z} = R_0(x, y) + \dots + R_s(x, y) z^s . \end{cases}$$

On setting

$$(11.8) \quad x = x_0 + \varepsilon^p t , \quad y = y_0 + \varepsilon^q u , \quad z = \varepsilon^{-r} v ,$$

one obtains

$$(11.9) \quad \begin{cases} \varepsilon^{q-p+r} \frac{du}{dt} = v , \\ \varepsilon^{(s-1)r-p} \frac{dv}{dt} = R_s(x_0 + \varepsilon^p t, y_0 + \varepsilon^q u) v^s + O(\varepsilon) . \end{cases}$$

When $q-p+r = 0$, $(s-1)r - p = 0$, $r = 1$, i.e., $p = s - 1$, $q = s-2$, one finds the reduced system

$$\frac{du}{dt} = v , \quad \frac{dv}{dt} = R_s(x_0, y_0) v^s ;$$

for this system to be stable, it is necessary, according to theorem II, that $s \leq 2$.

Therefore, in order that equation (11.1) be stable, it is necessary that $R(x, y, \dot{y})$ be, with respect to \dot{y} , a polynomial of the second degree.

For convenience, we shall write equation (11.1) as

$$(11.10) \quad \ddot{y} = A(x, y) \dot{y}^2 + B(x, y) \dot{y} + C(x, y) ,$$

where A, B, C are rational functions of y with coefficients analy-

tic in x .

12. The next step in our investigation is to characterize the functions A, B, C regarded as functions of y . Precisely , we shall show that

- i. All the poles of $A(x, y)$ regarded as a function of y are simple ;
- ii. The poles of $B(x, y)$ and $C(x, y)$ regarded as functions of y are included among those of $A(x, y)$ and are simple ;
- iii. The degrees of the polynomials $\frac{B(x, y)}{A(x, y)}$ and $\frac{C(x, y)}{A(x, y)}$ in y are at most 4 and 6 , respectively .

Proof of i . Set $x = x_0 + \varepsilon t$, where x_0 is arbitrary ; equation (11.10) becomes

$$\frac{d^2 y}{dt^2} = A(x_0 + \varepsilon t, y) \left(\frac{dy}{dt} \right)^2 + O(\varepsilon) .$$

In order that the reduced equation

$$\frac{d^2 y}{dt^2} = A(x_0, y) \left(\frac{dy}{dt} \right)^2$$

be stable , it is necessary that $A(x_0, y)$ be identical to one of the nine types enumerated above [see § 9 , Table II] . Because x_0 is arbitrary , the a, b, c, d, e, f, g of these nine types are now to be regarded as analytic functions of x ; these functions may eventually be identically zero or constants .

Now , set

$$u = y, \quad u = \frac{1}{ay + b}, \quad u = \frac{cy + d}{ay + b},$$

$$u = \frac{af - be}{cf - de} \cdot \frac{cy + d}{ay + b}$$

according as $A(x,y)$ coincides respectively with the first type of § 9, Table II, or the second, or the third, or the fourth, or with one of the other types; then, one finds that $A(x,y)$ may be assumed to be one of the following eight distinct rational functions

Table III

- i. 0 ;
- ii. $\frac{1}{y}$;
- iii. $(1 - \frac{1}{n}) \frac{1}{y}$, $n > 1$ an integer ;
- iv. $\frac{1}{2y} + \frac{1}{y-1}$;
- v. $\frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right)$;
- vi. $\frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right)$;
- vii. $\frac{2}{3} \cdot \frac{1}{y} + \frac{1}{2} \cdot \frac{1}{y-1}$;
- viii. $\frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-H} \right)$.

In viii, H may be a constant ($\neq 0, 1$) or may depend on x ; in the latter case, on taking a new variable $t = H(x)$,

one may suppose that $H(x)$ coincides with x .

[As an example, we verify the result for $u = \frac{1}{a y + b}$,

where a and b are analytic functions of x ; then,

$$y = \frac{1}{a} \left(\frac{1}{u} - b \right),$$

$$\dot{y} = -\frac{1}{a} \frac{\dot{u}}{u^2} + M(x, u),$$

$$\ddot{y} = -\frac{1}{a} \left(\frac{\ddot{u}}{u^2} - \frac{2\dot{u}^2}{u^3} \right) + N(x, u) \dot{u} + P(x, u),$$

where M , N , P are rational functions of u with coefficients analytic in x .

Therefore, the equation

$$\ddot{y} = \frac{\dot{y}^2}{ay + b} + B(x, y) \dot{y} + C(x, y)$$

assumes the form

$$\ddot{u} = \frac{\dot{u}^2}{u} + B_1(x, u) \dot{u} + C_1(x, u),$$

where B_1 , C_1 are rational functions of u with coefficients analytic in x .

Note also that the third type of § 9, Table II is now considered as a special case of the fourth type (for $n = \infty$).

Proof of ii. Let $y = a(x)$ be a pole of order j of $A(x, y)$, of order k of $B(x, y)$, and of order ℓ of $C(x, y)$. One has $j = 0$ or 1 ; we suppose that at least one of the integers k and ℓ is greater than j . On replacing $y + a(x)$ by y , one may

assume $a(x) \equiv 0$.

In a neighborhood of the pole $y = 0$, equation (11.10) may be written as

$$(12.1) \quad \ddot{y} = \frac{\dot{y}^2}{y} \left[1 - \frac{1}{n} + o(y) \right] + \frac{\dot{y}}{y^k} \left[B(x, 0) + o(y) \right] \\ + \frac{1}{y^\ell} \left[C(x, 0) + o(y) \right],$$

where $n \neq 0$ is a positive or negative integer which eventually may be infinite. Note that $n = 1$ when $j = 0$.

Now set $y = \varepsilon u$ and

- i. $x = x_0 + \varepsilon^k t$ when $\ell \leq 2k - 1$,
- ii. $x = x_0 + \varepsilon^{(1+\ell)/2} t$ when $\ell > 2k - 1$.

Then, equation (12.1) becomes

$$\ddot{u} = \left(1 - \frac{1}{n} \right) \frac{\dot{u}^2}{u} + a \frac{\dot{u}}{u^k} + o(\varepsilon), \quad \text{when } \ell < 2k - 1,$$

$$\ddot{u} = \left(1 - \frac{1}{n} \right) \frac{\dot{u}^2}{u} + \frac{b}{u} + o(\varepsilon), \quad \text{when } \ell > 2k - 1,$$

$$\ddot{u} = \left(1 - \frac{1}{n} \right) \frac{\dot{u}^2}{u} + a \frac{\dot{u}}{u} + \frac{b}{u^\ell} + o(\varepsilon), \quad \text{when } \ell = 2k - 1;$$

here, $a \equiv B(x_0, 0)$, $b \equiv C(x_0, 0)$.

The reduced equations are contained in

$$(12.2) \quad \ddot{u} = \left(1 - \frac{1}{n} \right) \frac{\dot{u}^2}{u} + a \frac{\dot{u}}{u^k} + \frac{b}{u^{2k-1}}$$

[if $a \neq 0$, k is necessarily a positive integer; if $a = 0$, one has $2k-1 \equiv \ell$, where ℓ is a positive integer].

In order that equation (12.2) be stable , it is necessary, according to theorem V , that $k = 1$ and $n \neq 1$. This proves ii.

Proof of iii . The result ii is essential for it enables one to give upper bounds for the degrees of B and C . Indeed, let $D(x,y)$ be the least common denominator of the partial fractions of $A(x,y)$, i.e. (see Table III) ,

$$\begin{aligned} D(x,y) &= y && \text{for ii , iii} , \\ &= y(y - 1) && \text{for iv - vii} , \\ &= y(y - 1)(y - H) && \text{for viii} . \end{aligned}$$

Then , one may write

$$B(x,y) = \frac{M(x,y)}{D(x,y)} , \quad C(x,y) = \frac{N(x,y)}{D(x,y)} ,$$

where $M(x,y)$ and $N(x,y)$ are polynomials in y of degrees m and n , respectively , with coefficients analytic in x .

On making the substitution $y = u^{-1}$, equation (11.10) becomes

$$\ddot{u} = \frac{\dot{u}^2}{u} \left[2 - \frac{A\left(x, \frac{1}{u}\right)}{u} \right] + B\left(x, \frac{1}{u}\right) \dot{u} - C\left(x, \frac{1}{u}\right) u^2 .$$

Now , we consider two cases according as to whether $A = 0$ or $A \neq 0$.

i. $A = 0$. Then , $u = 0$ is a simple pole for the coefficient of \dot{u}^2 so that necessarily $m \leq 1$, $n \leq 3$.

ii. $A \neq 0$. On taking into account the various types of $A(x,y)$ given above (Table III , i-viii) , one sees that

$u = 0$ is still a simple pole of the coefficient of u^2 ;
therefore , if q is the degree of D in y , one has

$$m \leq q + 1 , \quad n \leq q + 3 .$$

Thus , $q \leq 4$, $m \leq 4$, $n \leq 6$. More precisely , if $A(x,y)$ coincides with one of the types i-viii (Table III) , the degrees (m,n) are (1,3) for i , (2,4) for ii and iii , (3,5) for iv-vii , and (4,6) for viii .

13. The above method enables one to obtain in a simple manner conditions of stability which restrict the form of the coefficients A , B , C of equations (11.10). To obtain other necessary conditions for stability , one must , according to Painlevé , introduce a parameter ϵ into the equation and develop the solution $y(x,\epsilon)$ of the new equation as a series in powers of ϵ . As we have seen , the coefficients of this development are determined by quadratures ; the conditions that their critical points are fixed are necessary conditions for stability . This method must be applied to all the poles of $A(x,y)$ and to $y = \infty$. [For $y = \infty$, one may set $y = \frac{1}{u}$ or use the transformation $y = u/\epsilon$] .

However , to avoid the numerous and arduous computations involved in the method of Painlevé , we shall use another , simpler method .

14. The values of y for which the general existence theorem of Cauchy does not apply are $y = \infty, 0, 1, H(\neq 0,1)$, or x (according to the type of the equation) . In the following paragraphs , we shall consider particularly the case $y = \infty$; the other cases may be settled directly or by appropriate substitutions of the independent variable .

In a neighborhood of $y = \infty$, one has

$$(14.1) \quad \begin{cases} \frac{M(x,y)}{D(x,y)} = a y + a_1 + \frac{M_0}{D} , \\ \frac{N(x,y)}{D(x,y)} = b y^3 + b_1 y^2 + b_2 y + b_3 + \frac{N_0}{D} , \end{cases}$$

where M_0, N_0 are polynomials in y , each degree of which is less than the degree of D , and where the a 's and b 's are analytic functions of x .

Further ,

$$(14.2) \quad A(x,y) = \frac{A}{y} + O(1) ,$$

where the constant A has the following values

$$(14.3) \quad \begin{cases} 0(\text{type i}) ; 1(\text{type ii}) ; 1 - \frac{1}{n}, n > 1 \text{ an integer} \\ \quad \quad \quad (\text{type iii}) ; \\ \frac{3}{2} (\text{types iv,vi,viii}) ; \frac{4}{3} (\text{type v}) ; \frac{7}{6} (\text{type vii}) . \end{cases}$$

These values correspond to $A = 1 - \frac{1}{n}$, n an integer ,

where n has the following values

$$(14.4) \quad \begin{cases} n = 1(\text{type i}) ; & n = \infty(\text{type ii}) ; & n > 1(\text{type iii}); \\ n = -2(\text{types iv,vi,viii}); & n = -3(\text{type v}); & n = -6(\text{type vii}). \end{cases}$$

Then , equation (11.10) may be rewritten as

$$(14.5) \quad \ddot{y} = \left[A + O(y) \right] \frac{\dot{y}^2}{y} + \left[ay + a_1 + O(y) \right] \dot{y} + by^3 + b_1 y^2 + O(y).$$

Let us now determine necessary conditions for the absence of parametric critical points for the equation (14.5). To do this , suppose that in a neighborhood of $x = x_0$, one has

$$(14.6) \quad y(x) = \frac{s(x)}{(x - x_0)^r} ,$$

where $r > 0$ and $s(x_0) \neq 0$; $s(x)$ is a holomorphic function of x .

Substitute $y(x)$ given by (14.6) into (14.5) and note that

$$\dot{y}(x) = - \frac{r s(x_0)}{(x-x_0)^{r+1}} \left[1 + O(x - x_0) \right] ,$$

$$\ddot{y}(x) = \frac{r(r+1) s(x_0)}{(x - x_0)^{r+2}} \left[1 + O(x - x_0) \right] .$$

First , suppose that $a(x_0)$ or $b(x_0)$ is not zero ; then , the dominant terms arise from \ddot{y} , $\frac{\dot{y}^2}{y}$, $y \dot{y}$, y^3 and are

respectively proportional to

$$(x-x_0)^{-r-2} , \quad (x-x_0)^{-r-2} , \quad (x-x_0)^{-2r-1} , \quad (x-x_0)^{-3r} .$$

Therefore , to obtain an identity at least two of the numbers

$r+2$, $2r+1$, $3r$ must be equal ; this gives $r = 1$.

Second , suppose that $a(x_0)$ and $b(x_0)$ are both zero ; then the dominant terms arise from \ddot{y} , $\frac{\dot{y}^2}{y}$, \dot{y} , y^2 and are respectively proportional to

$(x-x_0)^{-r-2}$, $(x-x_0)^{-r-2}$, $(x-x_0)^{-r-1}$, $(x-x_0)^{-2r}$
so that $r = 2$.

15. Suppose $r = 1$ and set

$$(15.1) \quad y = \frac{s(x)}{z} \quad , \quad \dot{z} = 1 + u z .$$

For convenience and for future reference , we note the following formulas

$$(15.2) \quad \dot{y} = - \frac{s}{z^2} - \frac{s}{z} \left(u - \frac{\dot{s}}{s} \right) ,$$

$$(15.3) \quad \ddot{y} = \frac{2s}{z^3} + \frac{s}{z^2} \left(3u - 2 \frac{\dot{s}}{s} \right) - \frac{s}{z} \left(\dot{u} - u^2 - \frac{\ddot{s}}{s} + \frac{2\dot{s}}{s} u \right) ,$$

$$(15.4) \quad y \dot{y} = - \frac{s^2}{z^3} - \frac{s^2}{z^2} \left(u - \frac{\dot{s}}{s} \right) ,$$

$$(15.5) \quad \frac{\dot{y}}{y} = - \frac{1}{z} - u + \frac{\dot{s}}{s} ,$$

$$(15.6) \quad \ddot{y} - \left(1 - \frac{1}{n} \right) \frac{\dot{y}^2}{y} = \frac{s}{z^3} \left(1 + \frac{1}{n} \right) + \frac{s}{z^2} \left[\left(1 + \frac{z}{n} \right) u - \frac{2}{n} \frac{\dot{s}}{s} \right] \\ - \frac{s}{z} \left[\dot{u} + \frac{2}{n} \frac{\dot{s}}{s} u - \frac{u^2}{n} - \frac{\ddot{s}}{s} + \left(1 - \frac{1}{n} \right) \frac{\dot{s}^2}{s^2} \right] ;$$

in addition ,

$$(15.7) \quad A(x,y) = A \frac{z}{s} + O(z^2) \quad ;$$

$$(15.8) \quad \frac{M}{D} = a \frac{s}{z} + O(1) \quad \text{when } a \neq 0 \quad ,$$

$$= a_1 + O(z) \quad \text{when } a = 0 \quad ;$$

$$(15.9) \quad \frac{N}{D} = b \frac{s^3}{z^3} + O\left(\frac{1}{z^2}\right) \quad \text{when } b \neq 0 \quad ,$$

$$= b_1 \frac{s^2}{z^2} + O\left(\frac{1}{z}\right) \quad \text{when } b = 0 \quad .$$

Set

$$(15.10) \quad P = b s^2 - a s + A + 2 \quad ,$$

$$(15.11) \quad p = 3 - 2A + a s \quad .$$

Introduce $y(x)$ given by (15.1) into the equation (14.5) and rewrite the result as

$$(15.12) \quad z \dot{u} = - \frac{P}{z} + p u + h(x,z) + z H(x,z;u) \quad ,$$

where $h(x,z)$ and $H(x,z;u)$ are analytic functions of x and holomorphic functions of z in a neighborhood of $z = 0$; $H(x,z;u)$ is a polynomial in u .

Now , we determine s by setting $P = 0$; it then follows from theorem IV that a necessary condition for the equation (11.10) to be stable is that p be an integer .

These conditions will determine the possible values of a and b .

16. Consider the relations

$$(16.1) \quad b s^2 - a s - 1 - \frac{1}{n} = 0 \quad ,$$

$$(16.2) \quad p = 1 + \frac{2}{n} + a s \quad ,$$

where p must be an integer .

i. Suppose $b = 0$ and $a \neq 0$; then $a s + 1 + \frac{1}{n} = 0$ and

$p = \frac{1}{n}$. In order that p be an integer , one must have $n = 1$

or $n = \infty$, so that

$$(16.3) \quad n = 1 \quad , \quad p = 1 \quad , \quad a s + 2 = 0 \quad ;$$

$$(16.4) \quad n = \infty \quad , \quad p = 0 \quad , \quad a s + 1 = 0 \quad .$$

ii. Suppose $a = 0$ and $b \neq 0$; in order that $p = 1 + \frac{2}{n}$

be an integer , one must have $n = 2, 1, -2$ or ∞ . Therefore , the following cases arise

$$(16.5) \quad n = 1 \quad , \quad p = 3 \quad , \quad b s^2 - 2 = 0 \quad ;$$

$$(16.6) \quad n = 2 \quad , \quad p = 2 \quad , \quad 2 b s^2 - 3 = 0 \quad ;$$

$$(16.7) \quad n = -2 \quad , \quad p = 0 \quad , \quad 2 b s^2 - 1 = 0 \quad ;$$

$$(16.8) \quad n = \infty \quad , \quad p = 1 \quad , \quad b s^2 = 1 \quad .$$

iii. Suppose $a b \neq 0$. On eliminating s between (16.1) and (16.2) , one obtains

$$(16.9) \quad b \left(p - 1 - \frac{2}{n} \right)^2 = a^2 \left(p - \frac{1}{n} \right) \quad .$$

Taking into account the value of b given by (16.9) , one rewrites (16.1) as

$$\left(p - \frac{1}{n}\right) a^2 s^2 - \left(p-1-\frac{2}{n}\right)^2 as - \left(1+\frac{1}{n}\right)\left(p-1-\frac{2}{n}\right)^2 = 0.$$

This equation determines the product as ; its roots are

$$p - 1 - \frac{2}{n} \quad \text{and} \quad - \frac{(n+1)(np-n-2)}{np-1}$$

so that [cf. (16.2)]

$$1 + \frac{2}{n} + as = \begin{cases} p, \\ \frac{p+2+n}{pn-1} \equiv q. \end{cases}$$

Therefore , the integers p and q satisfy the Diophantine equation

$$(16.10) \quad p + q + n + 2 = p q n .$$

The integral solutions of (16.10) are given in Appendix I and determine all the distinct possibilities that we shall have to consider later on .

These possibilities are given below together with the corresponding relation (16.9) and the related values of p and s .

$$\underline{n = 1}$$

$$(16.11) \quad a = 0, \quad b \neq 0, \quad p = 3, \quad b s^2 - 2 = 0 ;$$

$$(16.12) \quad a \neq 0, \quad b = 0, \quad p = 1, \quad a s + 2 = 0 ;$$

$$(16.13) \quad 9b + a^2, \quad \begin{cases} p = 0, & as = -3, \\ p = -3, & as = -6 ; \end{cases}$$

$$(16.14) \quad b = a^2, \quad \begin{cases} p = 2, & as = -1, \\ p = 5, & as = 2. \end{cases}$$

$$\underline{n = \infty}$$

$$(16.15) \quad a = 0, \quad b \neq 0, \quad p = 1, \quad bs^2 - 1 = 0;$$

$$(16.16) \quad a \neq 0, \quad b = 0, \quad p = 0, \quad as + 1 = 0.$$

$$\underline{n \geq 2}.$$

$$(16.17) \quad b(n+2)^2 + na^2 = 0, \quad \begin{cases} p=0, \quad nas=-(n+2), \\ p=-(n+2), \quad nas=-(n+1)(n+2). \end{cases}$$

$$\underline{n = 2}.$$

$$(16.18) \quad 2b = a^2, \quad \begin{cases} p = 1, \quad as + 1 = 0, \\ p = 5, \quad as - 3 = 0; \end{cases}$$

$$(16.19) \quad a = 0, \quad b \neq 0, \quad p = 2, \quad 2bs^2 - 3 = 0.$$

$$\underline{n = 3}.$$

$$(16.20) \quad 3a^2 = 2b, \quad \begin{cases} p = 1, \quad 3as = -2, \\ p = 3, \quad 3as = 4. \end{cases}$$

$$\underline{n = 5}.$$

$$(16.21) \quad 5a^2 = b, \quad \begin{cases} p = 1, \quad 5as = -2, \\ p = 2, \quad 5as = 3. \end{cases}$$

$$\underline{n = -2}.$$

$$(16.22) \quad a = 0, \quad b \neq 0, \quad p = 0, \quad 2bs^2 = 1.$$

$$(16.23) \quad b = 3 a^2, \quad \begin{cases} p = 1, & 3 a s = 2, \\ p = 0, & 3 a s + 1 = 0. \end{cases} \quad \underline{n = -3}.$$

$$(16.24) \quad 8 b = 3 a^2, \quad \begin{cases} p = 0, & 3 a s = -2, \\ p = 4, & 3 a s = 10. \end{cases} \quad \underline{n = -6}.$$

17. Now suppose $r = 2$; then $a(x) \equiv 0$, $b(x) \equiv 0$. Set

$$(17.1) \quad y = \frac{t(x)}{z^2}, \quad \dot{z} = 1 + u z$$

so that

$$(17.2) \quad \dot{y} = -\frac{2t}{z^3} - \frac{t}{z^2} \left(2u - \frac{\dot{t}}{t} \right),$$

$$(17.3) \quad \ddot{y} = \frac{6t}{z^4} + \frac{2t}{z^3} \left(5u - 2\frac{\dot{t}}{t} \right) - 2\frac{t}{z^2} \left(\ddot{u} + 2\frac{\dot{t}}{t} u - \frac{\ddot{t}}{2t} - 2u^2 \right),$$

$$(17.4) \quad \ddot{y} - \left(1 - \frac{1}{n} \right) \frac{\dot{y}^2}{y} = \frac{2t}{z^4} \left(1 + \frac{2}{n} \right) + \frac{2t}{z^3} \left[\left(1 + \frac{4}{n} \right) u + \frac{2}{n} \frac{\dot{t}}{t} \right] \\ - \frac{2t}{z^2} \left[\ddot{u} + \frac{2\dot{t}}{nt} u - \frac{\ddot{t}}{2t} + \left(1 - \frac{1}{n} \right) \frac{\dot{t}^2}{2t^2} - \frac{2u^2}{n} \right].$$

Introduce $y(x)$ given by (17.1) into the equation (14.5)

and rewrite the result as (15.12) with

$$(17.5) \quad P = b_1 t - 2 - \frac{4}{n},$$

$$(17.6) \quad p = 1 + \frac{4}{n}.$$

Now , we determine $t(x)$ by setting $P = 0$. In order that the equation (11.10) be stable , it is necessary that p be an integer ; we suppose $p \gg 0$. The only five distinct possibilities are

$$(17.7) \quad n = \infty \quad , \quad p = 1 \quad , \quad b_1 t = 2 \quad ;$$

$$(17.8) \quad n = 1 \quad , \quad p = 5 \quad , \quad b_1 t = 6 \quad ;$$

$$(17.9) \quad n = 2 \quad , \quad p = 3 \quad , \quad b_1 t = 4 \quad ;$$

$$(17.10) \quad n = 4 \quad , \quad p = 2 \quad , \quad b_1 t = 3 \quad ;$$

$$(17.11) \quad n = -4 \quad , \quad p = 0 \quad , \quad b_1 t = 1 \quad .$$

When n is an arbitrary integer , one has also to consider the case $a \equiv b \equiv b_1 \equiv 0$.

18. We have also to take into account the values $y = 0$, 1 or x .

For $y = 0$ [resp. $y = 1$] , one may set $y = u^{-1}$ [resp. $y^{-1} = u^{-1}$] and use the results of the preceding paragraphs .

One may also proceed in a more direct manner . For instance , for $y = 0$, one may set

$$(18.1) \quad y = s z \quad , \quad \dot{z} = 1 + u z$$

so that

$$(18.2) \quad \begin{cases} \dot{y} = s + s z \left(u + \frac{\dot{s}}{s} \right) , \\ \ddot{y} = 2\dot{s} + s u + s z \left(\dot{u} + 2 \frac{\dot{s}}{s} u + \frac{\ddot{s}}{s} + u^2 \right) \end{cases}$$

and use the above method .

In what follows , we shall have to consider movable poles

[or zeros or unities] of order one or two ; to abbreviate , we use the obvious notations P_1 , P_2 [or Z_1 , Z_2 or U_1 , U_2] and for example , speak of an equation of class (P_1 , Z_1 , U_1) to signify an equation having movable poles , zeros and unities , all of the first order .

To continue our investigations , we must consider separately each of the eight types of equations corresponding to the eight possible values of $A(x,y)$ [see Table III , i-viii] .

Observe also that to obtain canonical forms for the stable equations , it is often most convenient to use a transformation [call it $T(\lambda , \mu , \varphi)$]

$$(18.3) \quad y(x) = \lambda(x) u + \mu(x) \quad , \quad t = \varphi(x)$$

which does not alter the main features of the equations considered [$\lambda(x)$, $\mu(x)$, $\varphi(x)$ are analytic functions of x ; in some cases , one has $\lambda(x) \equiv 1$, $\mu(x) \equiv 0$ or $\varphi(x) \equiv x$] .

For convenience , and future use , we note the following formulas where primes denote differentiations with

respect to t , i.e. $u' = \frac{du}{dt}$, $u'' = \frac{d^2u}{dt^2}$,

$$(18.4) \quad \begin{cases} \dot{y} = \dot{\lambda} u + \lambda \dot{\varphi} u' + \dot{\mu} \quad , \\ \ddot{y} = \ddot{\lambda} u + (2 \dot{\lambda} \dot{\varphi} + \lambda \ddot{\varphi}) u' + \lambda \dot{\varphi}^2 u'' + \ddot{\mu} \quad . \end{cases}$$

When $\mu = 0$, one finds

$$\ddot{y} - \frac{\dot{y}^2}{y} = \lambda \dot{\varphi}^2 \left(u'' - \frac{u'^2}{u} \right) + \left(\ddot{\lambda} - \frac{\dot{\lambda}^2}{\lambda} \right) u + \lambda \ddot{\varphi} u' \quad .$$

Throughout the remainder of this paper , we denote by $a, b, c, d,$

e, f, g, h, k, ℓ , with or without subscripts , analytic functions of x , (not always the same) and by H and K , with or without subscripts , (arbitrary) constants , (not always the same) . Moreover , we set $\Lambda = \frac{\dot{\lambda}}{\lambda}$, $\Phi = \frac{\ddot{\phi}}{\dot{\phi}}$ so that

$$(18.5) \quad \dot{\Lambda} = \frac{\ddot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} .$$

Appendix I .

The Diophantine equation $p + q + n + 2 = p q n$.

19. The problem of finding all the possible types of stable equations (11.1) depends on the problem of finding all the integral solutions of the Diophantine equation

$$(19.1) \quad p + q + n + 2 = p q n .$$

We suppose $n \geq 1$ or $n = -2, -3$ or -6 and $p \geq 0$.

1. $n = 1$. Equation (19.1) is

$$(19.2) \quad p + q + 3 = p q .$$

When $p = 0$, one has $q = -3$; when $p = 1$, equation (19.2) has no solution.

Suppose $p \geq 2$. Equation (19.1) may be rewritten as

$$(19.3) \quad \frac{1}{q} \left(1 + \frac{3}{p} \right) = 1 - \frac{1}{p}$$

so that $q > 0$; we may even suppose $q \geq 2$ because when $q = 1$, equation (19.3) has no solution.

Since

$$\left(1 - \frac{1}{q} \right) p = 1 + \frac{3}{q} ,$$

one finds $p \leq 5$; there are four cases to consider, i.e. $p = 2, 3, 4, 5$.

It is readily seen that for $p = 4$, equation (19.2) has no integral solution. There remain the three distinct solutions

$p = 2, q = 5; p = 3, q = 3; p = 5, q = 2.$

Therefore, when $n = 1$, the only integral solutions (p, q) of (19.2) are

(19.4) $(0, -3), (2, 5), (3, 3), (5, 2).$

ii. $n = \infty$. Equation (19.1) reduces to $p q = 1$ and has the only integral solution $p = 1, q = 1$.

iii. $n \geq 2$. For $p = 0$, we have $q = -(n+2)$.

Suppose $p > 0$; equation (19.1) rewritten as

$$\frac{1}{q} \left(\frac{1}{p} + \frac{1}{n} + \frac{2}{p n} \right) = 1 - \frac{1}{p n} > 0$$

shows that $q \geq 1$.

From (19.1), it also follows that

$$\frac{1}{q} + \frac{1}{n} + \frac{2}{q n} = \left(1 - \frac{1}{q n} \right) p;$$

since $n \geq 2, q \geq 1, q n \geq 2$, one has $p \leq 5$. Thus, we have to consider five possibilities, namely $p = 1, 2, 3, 4, 5$.

a. $p = 1$; the equation (19.1) is now $\frac{1}{n} + \frac{1}{q} + \frac{3}{n q} = 1$ and has been considered in i; its only distinct integral solutions are

$(n, q) = (\infty, 1), (5, 2), (3, 3), (2, 5).$

b. $p = 2$; equation (19.1) becomes $\frac{1}{n} + \frac{1}{q} + \frac{4}{n q} = 2$

and has only the two integral solutions

$(n, q) = (5, 1), (2, 2).$

c. $p = 3$; equation (19.1) becomes $\frac{1}{n} + \frac{1}{q} + \frac{5}{n q} = 3$

and has only the integral solution $(n,q) = (3,1)$.

d. $p = 4$; equation (19.1) has no integral solution .

e. $p = 5$; equation (19.1) is $\frac{1}{n} + \frac{1}{q} + \frac{7}{qn} = 5$

and has only the integral solution $(n,q) = (2,1)$.

iv. $n = -2$; equation (19.1) is $p + q + 2pq = 0$ and its only integral solution is $p = 0$, $q = 0$.

v. $n = -3$; equation (19.1) is $p + q + 3pq = 1$ and has only the two integral solutions $p = 1$, $q = 0$; $p = 0$, $q = 1$.

vi. $n = -4$; equation (19.1) is $p + q + 4pq = 2$ and has only the two integral solutions $p = 0$, $q = 2$; $p = 2$, $q = 0$.

vii. $n = -6$; equation (19.1) is $p + q + 6pq = 4$ and has only the two integral solutions $p = 0$, $q = 4$; $p = 4$, $q = 0$.

We summarize the distinct integral solutions of (19.1) ; we give the values of n and the corresponding values of (p,q) .

n	(p,q)
∞	1 , 1
1	2 , 5 ; 3 , 3
2	1 , 5 ; 2 , 2
3	1 , 3
5	1 , 2
-2	0 , 0
-3	0 , 1
-4	0 , 2
-6	0 , 4
$n \geq 2$	0 , $-(n+2)$

V. Equations of the type $A(x,y) = 0$.

20. Set

$$(20.1) \quad F(x,y) = e \dot{y} + e y^2 + f y + g .$$

The stable equations of this type are of the form

$$(20.2) \quad \ddot{y} = F(x, y) ,$$

$$(20.3) \quad \ddot{y} = a y \dot{y} + F(x, y) ,$$

$$(20.4) \quad \ddot{y} = a y \dot{y} - \frac{a^2}{9} y^3 + F(x, y) ,$$

$$(20.5) \quad \ddot{y} = b y^3 + F(x, y) ,$$

$$(20.6) \quad \ddot{y} = a y \dot{y} + a^2 y^3 + F(x, y) .$$

Equation (20.2) has parametric poles of the second order (P2) ; the other equations have movable poles of the first order (P1) . Therefore , the stable equations of the type $A(x,y) \equiv 0$ may be restricted to five distinct equations which may be represented symbolically by

$$P2 ; \quad (P1, p = 1) \quad ; \quad (P1 ; p = 0, p = -3) \quad ;$$

$$(P1 ; p = 3, p = 3) \quad ; \quad (P1 ; p = 2, p = 5) \quad .$$

Equations (20.2-6) belong to the type

$$(20.7) \quad \ddot{y} = a y \dot{y} + b y^3 + F(x, y) .$$

The general transformation [see (18.3)]

$$(20.8) \quad y = \lambda(x) u + \mu(x) , \quad t = \varphi(x)$$

does not alter the form of equation (20.7) which becomes

$$u'' = A u u' + B u^3 + C u' + E u^2 + F u + G ,$$

where A, B, C, E, F, G are given by

$$\begin{aligned}
 (20.9) \quad \dot{\varphi} A &= a \lambda , & \dot{\varphi}^2 B &= b \lambda^2 , \\
 \dot{\varphi} C &= (c + a\mu - 2\lambda - 2\dot{\varphi}) , \\
 \dot{\varphi}^2 E &= [a \dot{\lambda} + 3b \lambda \mu + e \lambda] , \\
 \dot{\varphi}^2 F &= a(\lambda \mu + \dot{\mu}) + 2e\mu + c\lambda + f + 3b\mu^2 - \frac{\ddot{\lambda}}{\lambda} , \\
 \lambda \dot{\varphi}^2 G &= a\mu \dot{\mu} + b\mu^3 + c\dot{\mu} + e\mu^2 + f\mu + g - \ddot{\mu} .
 \end{aligned}$$

21. Equation (20.2). We explain the particulars of our method on this equation which is of class P2.

According to section 17, set

$$(21.1) \quad y = \frac{s(x)}{z^2} , \quad \dot{z} = 1 + u z .$$

For equation (20.2), one has $n = 1$ and $et = 6$, $p = 5$ [see (17.8)] .

It follows from (20.9) that a transformation $y = \lambda u$ may be chosen so that $s = 1$, i.e. $e = 6$; equation (20.2) gives

$$(21.2) \quad z \dot{u} = 5u + c - \frac{f}{2} z - \frac{g}{2} z^3 + c u z + 2 z u^2 .$$

To apply our main theorem of stability (theorem IV), we must set [see section 6]

$$(21.3) \quad u = P(x; z) + z^5 v ,$$

where P is a polynomial in z of degree 4. To determine its coefficients, we derive from (21.2), (21.3), five relations; a sixth relation giving the condition for stability is then

deduced from theorem IV .

To simplify the problem , it is most convenient to transform equation (20.2) by a general transformation $T(\lambda, \mu, \varphi)$ into an equation of the same form for which $c = f = 0$. This is always possible , for one has to determine λ, μ, φ so as to satisfy [see (20.9)] $E = 6, C = 0, F = 0$; then λ, φ are given by

$$\frac{\dot{e}}{e} + \Lambda - 2\Phi = 0, \quad e - 2\Lambda - \Phi = 0$$

and are determined by quadratures ; μ is determined by

$$2e\mu + c\Lambda + f - \frac{\ddot{\lambda}}{\lambda} = 0.$$

With these simplifications , our problem is readily solved .

Indeed , equation (20.2) becomes

$$\ddot{y} = 6y^2 + g(x)$$

and (21.2) reduces to

$$z \dot{u} = 5u - \frac{g}{2} z^3 + 2zu^2.$$

Now , set $u = \alpha z^3 + \beta z^4 + z^5 v$ and determine α and β by

$$4\alpha = g, \quad 4\beta = \dot{g}; \quad \text{the condition for stability is then } \dot{\beta} = 0$$

(theorem IV) so that $\ddot{g}(x) = 0$ or $g(x) = Kx + H$.

The stable equations (20.2) are thus reducible to

$$(21.4) \quad \ddot{y} = 6y^2 + Kx + H.$$

By trivial changes of variables , this equation may be brought into one of the following canonical forms

i. when $K = H = 0$,

$$(21.5) \quad \ddot{y} = 6 y^2 ;$$

ii. when $K = 0$, $H \neq 0$,

$$(21.6) \quad \ddot{y} = 6 y^2 + \frac{1}{2} ;$$

iii. when $K \neq 0$,

$$(21.7) \quad \ddot{y} = 6 y^2 + x .$$

Equations (21.5) and (21.6) may be integrated by means of elliptic functions so that their general integral is a one-valued function of x . Equation (21.7) is not integrable in terms of classical transcendents ; its solution is one of the new transcendents discovered by P. Painlevé . (see table I ; equation I).

22. We consider now equations (20.3-6) all of class P1 .

Equation 20.3. Set $y = \frac{s(x)}{z}$ where $s(x)$ is given by

$as + 2 = 0$; note that $p = 1$. A transformation $T(\lambda, \mu, \varphi = x)$ may be chosen so as to secure $a = -2$ (i.e. $s = 1$) , and $e \equiv c$.

Then , one has

$$z \dot{u} = u - f z - g z^2 + c u z + z u^2 .$$

On setting $u = z v$, one obtains the condition for stability $f = 0$. The stable equations (20.3) are thus reducible to

$$(22.1) \quad \ddot{y} = -2 y \dot{y} + c(\dot{y} + y^2) + g$$

and are equivalent to the differential system

$$\dot{z} = 1 + z^2 v , \quad \dot{v} = c v - g .$$

The general solution of (22.1) is given by

$$w = \dot{y} + y^2 ,$$

where

$$\dot{w} = c w + g .$$

Therefore , w is determined by quadratures and y is given by a Riccati equation ; equation (22.1) is thus stable .

23. Equation (20.4). For this equation , $p = 0$, $as = -3$; $p = -3$, $as = -6$. We have only to consider $p = 0$ since a negative p does not result in a condition of stability . Through a transformation $T(\lambda)$, one may assume $a = -3$ and $s = 1$. The associated equation in \dot{u} is

$$z \dot{u} = c - e - f z - g z^2 + c z u + z u^2$$

which gives the condition for stability $c = e$.

The stable equations (20.4) are of the form

$$(23.1) \quad \ddot{y} = -3 y \dot{y} - y^3 + c(\dot{y} + y^2) + f y + g .$$

On setting $y = \frac{v}{v}$, this equation reduces to the linear equation of the third order

$$(23.2) \quad v - c \ddot{v} - f \dot{v} - g v = 0$$

which was investigated by E. Vessiot (Ann. Fac. Toulouse 1895, F2).

24. Equation (20.5). We have $p = 3$, $b s^2 = 2$. A transformation $T(\lambda, \mu, \varphi)$ may be chosen so that $b = 2$ (i.e. $s = \pm 1$) and $e = c = 0$.

Indeed , one has to determine λ, μ, φ by (set again

$$\frac{\dot{\lambda}}{\lambda} = \Lambda, \quad \frac{\ddot{\varphi}}{\varphi} = \Phi),$$

$$b \lambda^2 = 2 \dot{\varphi}^2, \quad c - 2\Lambda - 2\Phi = 0, \quad 3b\mu + e = 0;$$

λ and μ are given by quadratures through

$$\frac{\dot{b}}{b} + 2\Lambda - 2\Phi = 0, \quad c - 2\Lambda - 2\Phi = 0.$$

With the values assumed for b, c, e , equation (20.5) becomes

$$(24.1) \quad \ddot{y} = 2y^3 + fy + g;$$

its associated equation in \dot{u} is

$$(24.2) \quad z \dot{u} = 3u - fz - gs z^2 + z u^2.$$

Now, set

$$u = \beta z + \gamma z^2 + z^3 v$$

and determine β and γ by

$$2\beta = f, \quad \dot{\beta} = \gamma - gs;$$

the condition for stability is $\dot{\gamma} = 0$. Therefore, $\ddot{f} + 2\dot{g}s = 0$;

because $s = \pm 1$, this relation splits up into $\ddot{f} = 0, \dot{g} = 0$.

Hence, $f = Kx + H$; g is also an arbitrary constant (say K_1).

The stable equations (20.5) are thus reducible to

$$(24.3) \quad \ddot{y} = 2y^3 + (Kx + H)y + K_1.$$

When $K = 0$, this equation becomes

$$(24.4) \quad \ddot{y} = 2y^3 + Hy + K_1$$

and is at once integrable in terms of elliptic functions.

When $K \neq 0$, a trivial change of variable brings (24.3) into the canonical form

$$(24.5) \quad \ddot{y} = 2y^3 + xy + K_2.$$

This equation is not integrable in terms of classical transcendents ; its solution is one of the new transcendents discovered by P. Painlevé (see table I ; equation II) .

25. Equation (20.6). One has $p = 2$, $as = -1$ and $p = 5$, $as = 2$. A transformation $T(\lambda, \mu, \varphi)$ may be chosen so that $a = -1$, $e = c = 0$. Indeed , one has to determine λ, μ, φ by

$$\begin{aligned} a\lambda &= -\dot{\varphi} \quad , \quad c + a\mu - 2\lambda - \Phi = 0 \quad , \\ a\lambda + 3b\mu + e &= 0 \end{aligned}$$

or by

$$\frac{\dot{a}}{a} + \lambda - \Phi = 0 \quad , \quad c + a\mu - 2\lambda - \Phi = 0 \quad , \quad e + a\lambda + 3b\mu = 0 \quad ;$$

these equations have one and only one solution .

Equation (20.6) assumes the form

$$(25.1) \quad \ddot{y} = -y\dot{y} + y^3 + fy + g \quad ;$$

the associated equation in u is

$$(25.2) \quad z\dot{u} = (3-s)u - fz - \frac{g}{s}z^2 + zu^2 \quad .$$

a. First , consider $p = 2$, $s = 1$; the equation (25.2) becomes

$$(25.3) \quad z\dot{u} = 2u - fz - gz^2 + zu^2 \quad .$$

On setting $u = \beta z + z^2 v$, one obtains the relations

$$\beta = f \quad , \quad \dot{\beta} = -g$$

and hence the condition for stability $\dot{f} + g = 0$.

To simplify the notations , we set

$$f = -12V \quad , \quad g = 12\dot{V} \quad .$$

b. It follows from a , that the stable equations (25.1) are of the form

$$(25.5) \quad \ddot{y} = -y \dot{y} + y^3 - 12 V y + 12 \dot{V}.$$

Now , consider $p = 5$, $s = -2$; the condition for the stability of (25.5) gives rise to a condition on V .

The equation in u associated with (25.5) is

$$z \dot{u} = 5 u + 12 V z + 6 \dot{V} z^2 + z u^2 .$$

On setting

$$u = \beta z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + z^5 v ,$$

one obtains the relations

$$\begin{aligned} 4 \beta &= -12 V , & \dot{\beta} &= 6 \dot{V} + 3 \gamma , \\ \dot{\gamma} &= 2 , & \dot{\delta} + \beta \gamma &= \varepsilon , \\ \dot{\varepsilon} + \gamma^2 + 2 \beta \delta &= 0 ; \end{aligned}$$

these relations easily give

$$\begin{aligned} \beta &= -3 V , & \gamma &= -3 \dot{V} , & \delta &= -\frac{3}{2} \ddot{V} , \\ \varepsilon &= -\frac{3}{2} \ddot{V} + 9 V \dot{V} \end{aligned}$$

and hence the condition for stability

$$V^{(4)} - 12 \dot{V}^2 - 12 V \ddot{V} = 0$$

or

$$\frac{d^2}{dx^2} (\ddot{V} - 6 V^2) = 0$$

and finally

$$(25.6) \quad \ddot{V} = 6 V^2 + K x + H .$$

Thus , the stable equations (25.1) are of the form (25.5), where V is a solution of (25.6) , i.e. , an elliptic function

or a solution of the irreducible Painlevé's equation I.

c. That the equation (25.5) where V is given by (25.6) , is really a stable equation remains to be shown . To do this , we employ a method which shall often be used in what follows.

Observe that the stability condition corresponding to $p = 2$ (see a) is readily obtained and brings equation (25.1) into the form (25.5) . Moreover , one has

$$y = \frac{1}{z} \quad , \quad \dot{z} = 1 + z u \quad , \quad u = - 12 V z + z^2 v$$

so that $\frac{\dot{y}}{y} = - \frac{1}{z} + u$; it then follows that $\frac{\dot{y}}{y} + y = u$ is regular at the parametric poles corresponding to $p = 2$ and has simple poles at the parametric poles corresponding to $p = 5$.

Accordingly , set

$$(25.7) \quad \dot{y} + y^2 = v \quad ;$$

v has double poles at the parametric poles corresponding to $p = 5$; it is thus expected that the stability condition for V is connected with equation (20.1) .

Indeed , the equation (25.5) is equivalent to the differential system

$$(25.8) \quad \begin{cases} \dot{y} + y^2 = v \quad , \\ \dot{v} = v y - 12 V y + 12 \dot{V} \quad . \end{cases}$$

Hence , v is a rational function of y , \dot{y} and , vice versa , y is a rational function of v , \dot{v} ; therefore , if y is stable , v is also stable and conversely .

On eliminating y between the relations (25.8) , one finds

$$(25.9) \quad \ddot{v} = v^2 - 12 V \dot{v} + 12 \dot{V}$$

[write $\frac{\dot{y}}{y} + y = \frac{v}{y}$, and use the logarithmic derivatives
of $y = \frac{\dot{v} - 12 \dot{V}}{v - 12 V}$] .

The transformation $v - 6 V = 6 w$ brings (25.9) into the canonical form [see § 21]

$$\ddot{w} = 6 w^2 + \ddot{V} - 6 V^2 ;$$

in order that this equation be stable , one must have [see (21.4)]

$$\ddot{V} = 6 V^2 + K x + H$$

and

$$(25.10) \quad \ddot{w} = 6 w^2 + K x + H .$$

Therefore , the general solution of

$$\ddot{y} = - y \dot{y} + y^3 - 12 V y + 12 \dot{V} ,$$

where V is a particular solution of (25.10) , is

$$(25.11) \quad y = \frac{\dot{w} - \dot{V}}{w - V} ;$$

w is a solution of (25.10) distinct from V .

26. To sum up , the following set of equations may be considered as a set of canonical equations of the type $A(x,y)=0$.

i. $\ddot{y} = 0$;

ii. (P2 ; $p = 5$)

a. $\ddot{y} = 6 y^2 + K$

integrable by elliptic functions ;

b. $\ddot{y} = 6 y^2 + x$

not integrable in terms of classical transcendents ;

iii. (P1 ; p = 1)

$$\ddot{y} = -2 y \dot{y} + c(\dot{y} + y^2) + g$$

integrable by quadratures ;

iv. (P1 ; p = 0)

$$\ddot{y} = -3 y \dot{y} - y^3 + c(\dot{y} + y^2) + f y + g$$

may be reduced to a linear equation of the third order ;

v. (P1 ; p = 3 , 3)

$$a. \quad \ddot{y} = 2 y^3 + K y + H$$

integrable by elliptic functions ;

$$b. \quad \ddot{y} = 2 y^3 + x y + K$$

not integrable in terms of classical transcendents ;

vi. (P1 ; p = 2 , 5)

$$\ddot{y} = -y \dot{y} + y^3 - 12 V_1 y + 12 \dot{V}_1 ,$$

where V_1 is a solution of ii , a or ii , b .

Integration :

$$y = \frac{\dot{V} - \dot{V}_1}{V - V_1} ,$$

where V and V_1 are distinct solutions of ii , a or ii , b .

In iii and iv , c , f , g are arbitrary analytic functions of x .

To obtain the most general stable equations of the type $A(x,y) = 0$, one has to use a general transformation $T(\lambda, \mu, \varphi)$, where λ , μ , φ are arbitrary analytic functions of x .

VI. Equations of the type $A(x,y) = \frac{1}{y}$.

27. The stable equations of this type are of the form

$$(27.1) \quad \ddot{y} - \frac{\dot{y}^2}{y} = a y \dot{y} + a_2 \frac{\dot{y}}{y} + b y^3 + \frac{b_4}{y} + F(x,y) ,$$

where

$$(27.2) \quad F(x,y) = a_1 \dot{y} + b_1 y^2 + b_2 y + b_3 ;$$

the a's and b's satisfy conditions to be given later on .

The substitution $y v = 1$ transforms the equation (27.1) into an equation of the same form , namely

$$(27.3) \quad \ddot{v} - \frac{\dot{v}^2}{v} = a_2 v \dot{v} + a \frac{\dot{v}}{v} - b_4 v^3 - \frac{b}{v} + a_1 \dot{v} - b_3 v^2 - b_2 v - b_1 ;$$

therefore , the stability conditions corresponding to the parametric zeros of $y(x)$ may be deduced from the stability conditions produced by the parametric poles , according to the table of equivalence

$$(y) : \quad a \quad a_1 \quad a_2 \quad b \quad b_1 \quad b_2 \quad b_3 \quad b_4$$

$$(v) : \quad a_2 \quad a_1 \quad a \quad -b_4 \quad -b_3 \quad -b_2 \quad -b_1 \quad -b .$$

The transformation $y = \lambda(x) u$, $t = \varphi(x)$ does not alter the form of equation (27.1) which becomes

$$u'' - \frac{u'^2}{u} = A u' u + A_2 \frac{u'}{u} + B u^3 + \frac{B_4}{u} + A_1 u' + B_1 u^2 + B_2 u + B_3 ,$$

where on setting $\Lambda = \frac{\dot{\lambda}}{\lambda}$, one has

$$\begin{aligned} \dot{\varphi} A &= a \lambda, \quad \dot{\varphi} A_1 = a_1 - \frac{\ddot{\varphi}}{\dot{\varphi}}, \quad \lambda \dot{\varphi} A_2 = a_2, \\ \dot{\varphi}^2 B &= b \lambda^2, \quad \dot{\varphi}^2 B_1 = b_1 \lambda + a \dot{\lambda}, \quad \dot{\varphi}^2 B_2 = b_2 + a_1 \lambda - \dot{\lambda}, \\ \lambda \dot{\varphi}^2 B_3 &= b_3 + a_2, \quad \lambda^2 \dot{\varphi}^2 B_4 = b_4. \end{aligned}$$

To simplify the problem, one may choose λ, φ so that $a_1 = b_2 = 0$. [One has only to determine λ, φ in order that $\ddot{\varphi} = a_1 \dot{\varphi}, \dot{\lambda} - a_1 \lambda - b_2 = 0$].

28. The general solution of (27.1) may have simple or double parametric poles and simple or double parametric zeros; we consider these cases separately.

1. Double parametric poles. One has $a = b = 0$. The transformation $y = \frac{s(x)}{z^2}$, $z = 1 + u z$, where s is given by $b_1 s = 2$,

changes (27.1) into

$$\begin{aligned} zu &= u + \frac{z}{2} \frac{d^2}{dx^2} \lg s + \frac{a_2}{s} z^2 - \frac{1}{2s} \left(a_2 \frac{\dot{s}}{sa} + b_3 \right) z^3 \\ &\quad - \frac{b_4}{2s^2} z^5 + \frac{a_2}{s} z^3 u. \end{aligned}$$

Because $p = 1$, one sets $u = z v$ and finds the condition for stability $\frac{d^2}{dx^2} \lg s = 0$ or $\frac{d^2}{dx^2} \lg b_1 = 0$ (note that $b_1 s = 2$)

and finally

$$(28.1) \quad b_1 = H_1 e^{K_1 x}.$$

ii. Double parametric zeros . One has $a_2 = b_4 = 0$; according to the table of equivalence , the condition for stability is

$$(28.2) \quad b_3 = H_2 e^{K_2 x} .$$

iii. Simple parametric poles . Set $y = \frac{s}{z}$, $\dot{z} = 1 + u z$, where s is given by $bs^2 - as - 1 = 0$ and $p = 1 + as$; equation (27.1) becomes

$$(28.3) \quad z\dot{u} = pu - b_1 s - as + Az - \left(a_2 \frac{\dot{s}}{s} + b_3 \right) \frac{z^2}{s} - \frac{b_4}{s^2} z^3 + \frac{a_2}{s} z^2 u$$

with

$$(28.4) \quad A = \frac{d^2}{dx^2} \lg s + \frac{a_2}{s} .$$

It follows from (16.15) and (16.16) that we have to consider two cases according as to whether $a = 0$, $b \neq 0$ or $a \neq 0$, $b = 0$.

a) $a \neq 0$, $b = 0$; then $p = 0$, $as + 1 = 0$. The condition for stability is $b_1 s + as = 0$ or

$$(28.5) \quad b_1 = \dot{a} .$$

b) $a = 0$, $b \neq 0$; then $p = 1$, $bs^2 = 1$. Set $u = \alpha + z v$ in (28.3) ; the condition for stability is given by

$$\alpha = b_1 s \quad , \quad \dot{\alpha} = A \quad ,$$

or by

$$(28.6) \quad \frac{d}{dx} (b_1 s) = A .$$

From $bs^2 = 1$, it follows $s = \epsilon b^{-1/2}$, $\epsilon = \pm 1$ and from (28.4) and (28.5) ,

$$\epsilon \frac{d}{dx} b_1 b^{-1/2} = - \frac{1}{2} \frac{d^2}{dx^2} \lg b + \epsilon a_2 b^{-1/2}$$

or

$$(28.7) \quad \frac{d}{dx} b_1 b^{-1/2} = a_2 b^{-1/2}, \quad \frac{d^2}{dx^2} \lg b = 0$$

so that

$$(28.8) \quad b = H e^{Kx}.$$

iv. Simple parametric zeros. From iii a, b and the table of equivalence, we easily obtain the following conditions for stability.

a) $a_2 \neq 0$, $b_4 = 0$; $p = 0$, $a_2 s + 1 = 0$. The condition for stability is

$$(28.9) \quad b_3 + \dot{a}_2 = 0.$$

b) $a_2 = 0$, $b_4 = 0$; $p = 1$, $b_4 s^2 + 1 = 0$. The condition for stability is given by [see (28.6)]

$$(28.10) \quad \frac{d}{dx} (b_3 s) + \frac{d^2}{dx^2} \lg s + \frac{a}{s} = 0.$$

29. It follows from the preceding paragraphs that the stable equations of the type $A(x, y) = \frac{1}{y}$, may be restricted to six distinct equations which may be represented symbolically by

$$(29.1) \quad \left\{ \begin{array}{l} (P2; Z2) \quad ; \quad (P2; Z1, p = 0 \text{ or } p = 1) \quad ; \\ (P1, p = 0 ; Z1, p = 0 \text{ or } p = 1) \quad ; \quad (P1, p = 1; Z1, p = 1). \end{array} \right.$$

Indeed , the other equations are deducible from one or the other of the equations (29.1) by setting $y = u^{-1}$.

We consider separately each of the equations (29.1) .

30. P2;Z2. One has $a = b = a_2 = b_4 = 0$. The conditions for stability are given by (28.1) for P2 and by (28.2) for Z2.

The equation is

$$(30.1) \quad \ddot{y} - \frac{\dot{y}^2}{y} = H_1 e^{K_1 x} y^2 + H_2 e^{K_2 x}$$

and is a particular case of the irreducible equation III (see table I).

31. P2;Z1, p = 0 . One has $a = b = b_4 = 0$, $a_2 \neq 0$. For P2 , the condition for stability is given by (28.1) . For Z1 , $p = 0$, this condition is given by (28.9) . Set $a_2 = q$, $b_3 = -\dot{q}$ and write the equation

$$(31.1) \quad \ddot{y} - \frac{\dot{y}^2}{y} = q \frac{\dot{y}}{y} + H e^{Kx} y^2 - \dot{q} .$$

Now , define $y = \lambda u$, where λ is given by $\lambda H e^{Kx} = 1$ and set

$r = \frac{q}{\lambda}$; then because $\dot{r} = \frac{\dot{q}}{\lambda} - \frac{\dot{\lambda}}{\lambda^2}$, (31.1) becomes

$$(31.2) \quad \ddot{u} - \frac{\dot{u}^2}{u} = r \frac{\dot{u}}{u} + u^2 - \dot{r} .$$

For convenience , set $r = \dot{R}$ and $v = \frac{\dot{u} + \dot{R}}{u}$; equation (31.2)

is equivalent to the differential system

$$\frac{\dot{u}}{u} + \frac{\dot{R}}{u} = v \quad , \quad \dot{v} = u .$$

On eliminating u between these relations, one finds $\ddot{v} + \dot{R} = v\dot{v}$ so that v is determined by the Riccati equation $\dot{v} + R = \frac{1}{2} v^2 H_1$.

Therefore, u is stable.

32. P2; Z1, p = 1. One has $a = b = a_2 = 0$, $b_4 \neq 0$.

For P2, the condition for stability is given by (28.1).

For Z1, $p = 1$, one has $s = b_4^{-1/2} \epsilon$, $\epsilon = \pm i$ so that (28.10) becomes

$$\epsilon \frac{d}{dx} b_4^{-1/2} b_3 - \frac{1}{2} \frac{d^2}{dx^2} \lg b_4 = 0$$

and

$$b_4 = H_4 e^{2K_2 x}, \quad b_3 = H_3 e^{K_2 x}.$$

Equation (31.1) is written

$$(32.1) \quad \ddot{y} - \frac{\dot{y}^2}{y} = H_1 e^{K_1 x} y^2 + H_3 e^{K_2 x} + \frac{H_4}{y} e^{2K_2 x}$$

and is a particular case of the irreducible equation III (see table I).

33. P1, p = 0; Z1, p = 0. One has $b = b_4 = 0$, $aa_2 \neq 0$.

The conditions for stability are $b_1 = \dot{a}$, $b_3 + \dot{a}_2 = 0$ [see (28.5) and (28.9)] so that the equation takes the form

$$(33.1) \quad \ddot{y} - \frac{\dot{y}^2}{y} = a y \dot{y} + a_2 \frac{\dot{y}}{y} + \dot{a} y^2 - \dot{a}_2$$

Integration. It is easily seen that $\dot{y} - a y^2 = v$ is regular for the poles P1, $p = 0$; the equation (33.1) gives

$$\dot{v} + \dot{a}_2 = \frac{\dot{y}}{y} (v + a_2)$$

or $v + a_2 = K y$. Therefore, equation (33.1) is equivalent to the differential system

$$\dot{y} - a y^2 = v, \quad v + a_2 = K y$$

so that y is determined by the Riccati equation

$$\dot{y} = a y^2 + K y - a_2$$

and is stable.

34. P1, $p = 0$; Z1, $p = 1$. For P1, one has $a \neq 0$, $b = 0$, as $+1 = 0$ and the condition for stability is $b_1 = \dot{a}$.

For Z1, one has $b_4 \neq 0$, $b_4 s^2 + 1 = 0$; a transformation $y \rightarrow \lambda y$ enables one to assume $b_4 = -1$ or $s = \pm 1$; the condition for stability is then $\dot{b}_3 + a = 0$.

For convenience, set $b_3 = q$ so that $a = -\dot{q}$, $b_1 = -\ddot{q}$; the equation then assumes the form

$$(34.1) \quad \ddot{y} - \frac{\dot{y}^2}{y} = -\dot{q} y \frac{\dot{y}}{y} - \frac{1}{y} - \ddot{q} y^2 + q.$$

The transformation $y u = 1$ brings this equation to

$$(34.2) \quad \ddot{u} - \frac{\dot{u}^2}{u} = -\dot{q} \frac{\dot{u}}{u} + u^3 - q u^2 + \ddot{q}$$

which belongs to the class (Z1, $p = 0$; P1, $p = 1$).

To integrate the equation (34.1), observe that

$$v = \frac{\dot{y}}{y} + \dot{q} y - \frac{1}{y} + q$$

is regular for the parametric poles and for the parametric

zeros corresponding to $s = 1$.

Equation (34.1) is equivalent to the differential system

$$\dot{y} + q y^2 - 1 + q y = v y, \quad y \dot{v} = v.$$

On eliminating y between these relations, one finds

$$\ddot{v} + v \dot{v} - q v - q \dot{v} = 0$$

and v is a solution of the Riccati equation

$$\dot{v} + \frac{1}{2} v^2 - q v = H.$$

Set $v = 2 \frac{\dot{w}}{w}$; then v is a solution of

$$\ddot{w} - q \dot{w} - \frac{H}{2} w = 0.$$

Therefore $y(x)$ is determined by

$$y = \frac{1}{w \dot{w}} \left[w \ddot{w} - \dot{w}^2 \right]$$

and is stable.

35. P1, $p = 1$; Z1, $p = 1$. One has $a = a_2 = 0$, $b b_4 \neq 0$.

According to (28.7) and (28.8) and the table of equivalence,

the conditions for stability are

$$\text{for P1} \quad b = H_1 e^{2K_1 x}, \quad b_1 = H_2 e^{K_1 x};$$

$$\text{for Z1} \quad b_4 = H_4 e^{2K_2 x}, \quad b_3 = H_3 e^{K_2 x}.$$

The equation is

$$(35.1) \quad \ddot{y} - \frac{\dot{y}^2}{y} = H_1 e^{2K_1 x} y^3 + H_2 e^{K_1 x} y^2 + H_3 e^{K_2 x} + \frac{H_4}{y} e^{2K_2 x}.$$

A transformation $y \rightarrow \lambda(x) y$, where λ is defined by

$\lambda^2 e^{(K_1 - K_2)x} = 1$ and m by $K_1 + K_2 = 2m$, brings equation (35.1) into the form

$$(35.2) \quad \ddot{y} - \frac{\dot{y}^2}{y} = e^{2mx} \left(H_1 y^3 + \frac{H_4}{y} \right) + e^{mx} \left(H_2 y^2 + H_3 \right).$$

Now, we consider two cases according as to whether $m = 0$ or $m \neq 0$.

1. $m = 0$. The equation (35.2) is

$$(35.3) \quad \ddot{y} - \frac{\dot{y}^2}{y} = H_1 y^3 + \frac{H_4}{y} + H_2 y^2 + H_3$$

and may be integrated by a process which may be considered as a method of "variation of parameters". As this method will be used often later on, we shall explain its particulars on equation (35.3).

The general solution of $\ddot{y} - \frac{\dot{y}^2}{y} = 0$ is $\dot{y} = u y$ where u is an arbitrary constant.

Suppose now that u is a function of y and substitute $\dot{y} = u y$ into (35.3); one obtains

$$u y = H_1 y^3 + H_2 y^2 + H_3 + \frac{H_4}{y}$$

or

$$u \frac{du}{dy} = H_1 y + H_2 + \frac{H_3}{y^2} + \frac{H_4}{y^3}.$$

Therefore,

$$u^2 = H_1 y^2 + 2 H_2 y - \frac{2 H_3}{y} - \frac{H_4}{y^2} + H_5.$$

Finally, $y(x)$ is given by

$$(35.4) \quad \dot{y}^2 = H_1 y^4 + 2 H_2 y^3 + H_5 y^2 - 2 H_3 y - H_4$$

and is an elliptic function of x ; thus , $y(x)$ is stable .

ii. $m \neq 0$. A linear transformation on x transforms equation (35.2) into

$$(35.5) \quad \ddot{y} - \frac{\dot{y}^2}{y} = e^{2x} \left(H_1 y^3 + \frac{H_4}{y} \right) + e^x \left(H_2 y^2 + H_3 \right) ;$$

this equation is not integrable in terms of classical transcendents and is equation III of Table I .

A transformation $t = \varphi(x) = \frac{e^{mx}}{m}$ brings equation (35.2) into the other canonical form

$$(35.6) \quad \ddot{y} - \frac{\dot{y}^2}{y} = - \frac{\dot{y}}{x} + H_1 y^3 + \frac{H_4}{y} + \frac{1}{mx} (H_2 y^2 + H_3)$$

[Note that $H_1 H_4 \neq 0$] .

36. To sum up , the following set of equations may be considered as a set of canonical equations of the type $A(x,y) = \frac{1}{y}$.

$$i. \quad \ddot{y} - \frac{\dot{y}^2}{y} = 0 ;$$

ii. (P2;Z1,p = 0)

$$\ddot{y} - \frac{\dot{y}^2}{y} = r \frac{\dot{y}}{y} + y^2 - \dot{r}$$

reducible to a Riccati equation ;

iii. (P1,p = 0;Z1,p = 0)

$$\ddot{y} - \frac{\dot{y}^2}{y} = a y \dot{y} + a_2 \frac{\dot{y}}{y} + \dot{a} y^2 - \dot{a}_2$$

reducible to a Riccati equation ;

iv. (P1,p = 0; Z1,p = 1)

$$\ddot{y} - \frac{\dot{y}^2}{y} = -\dot{q} y \dot{y} - \frac{1}{y} - \ddot{q} y^2 + q$$

reducible to two Riccati equations ;

$$v. \quad \ddot{y} - \frac{\dot{y}^2}{y} = \alpha y^3 + \frac{\delta}{y} + \beta y^2 + \gamma$$

integrable by elliptic functions ;

$$\ddot{y} - \frac{\dot{y}^2}{y} = e^{2x} \left(\alpha y^3 + \frac{\delta}{y} \right) + e^x \left(\beta y^2 + \gamma \right)$$

or

$$\ddot{y} - \frac{\dot{y}^2}{y} = -\frac{\dot{y}}{x} + \alpha y^3 + \frac{\delta}{y} + \frac{1}{m x} (\beta y^2 + \gamma)$$

where the constants α , β , γ , δ may eventually be zero ;

$m \neq 0$ is also a constant .

VII. Equations of the type $A(x,y) = \left(1 - \frac{1}{n}\right) \frac{1}{y} .$

37. The stable equations of this type are of the form

$$(37.1) \quad \ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = a_1 y \dot{y} + a_2 \frac{\dot{y}}{y} + b_1 y^3 + \frac{b_4}{y} + F(x,y) ,$$

where

$$(37.2) \quad F(x,y) = a_1 \dot{y} + b_1 y^2 + b_2 y + b_3 ;$$

the a's and b's satisfy conditions to be given later on .

The substitution $yv = 1$ transforms equation (37.1) into an equation of the same form , namely

$$(37.3) \quad \ddot{v} - \left(1 + \frac{1}{n}\right) \frac{\dot{v}^2}{v} = \left(a_2 v + a_1 + \frac{a}{v}\right) \dot{v} - b_4 v^3 - b_3 v^2 - b_2 v - b_1 - \frac{b}{v} ;$$

therefore , the conditions for stability corresponding to the parametric zeros of $y(x)$ may be deduced from the conditions for stability produced by the parametric poles , according to the table of equivalence

$$\begin{array}{l} (y) : \quad n \quad a \quad a_1 \quad a_2 \quad b \quad b_1 \quad b_2 \quad b_3 \quad b_4 \\ (v) : \quad -n \quad a_2 \quad a_1 \quad a \quad -b_4 \quad -b_3 \quad -b_2 \quad -b_1 \quad -b \end{array}$$

This table shows that without loss of generality , we may restrict n to be positive , and accordingly do so .

The transformation $y = \lambda(x) u$, $t = \varphi(x)$ does not alter the form of equation (37.1) which becomes

$$u'' - \left(1 - \frac{1}{n}\right) \frac{u'^2}{u} = \left(Au + A_1 + \frac{A_2}{u}\right) u' + Bu^3 + B_1u^2 + B_2u + B_3 + \frac{B_4}{u},$$

where

$$\dot{\varphi} A = a \lambda, \quad \dot{\varphi} A_1 = a_1 - \frac{2}{n} \lambda - \dot{\varphi}, \quad \lambda \dot{\varphi} A_2 = a_2,$$

$$\dot{\varphi}^2 B = b \lambda^2, \quad \dot{\varphi}^2 B_1 = b_1 \lambda + a \dot{\lambda},$$

$$\dot{\varphi}^2 B_2 = b_2 + a_1 \lambda - \lambda - \frac{1}{n} \lambda^2,$$

$$\lambda \dot{\varphi}^2 B_3 = b_3 + a_2 \lambda, \quad \lambda^2 \dot{\varphi}^2 B_4 = b_4.$$

For future use, we also note the following system of equations equivalent to equation (37.1); on setting $y = \frac{s}{z}$, $\dot{z} = 1 + uz$, one has

$$\begin{aligned} z \dot{u} &= \frac{1}{z} \left(1 + \frac{1}{n} + as - bs^2\right) + \left(1 + \frac{2}{n} + as\right) u \\ (37.4) \quad &+ a_1 - b_1 s - as - \frac{2}{n} \frac{\dot{s}}{s} + Az - \left(b_3 + a_2 \frac{\dot{s}}{s}\right) \frac{z^2}{s} - \frac{b_4 z^3}{s^2} \\ &+ \left(a_1 - \frac{2}{n} \frac{\dot{s}}{s}\right) z u + \frac{a_2}{s} z^2 u + \frac{z}{n} u^2 \end{aligned}$$

with

$$(37.5) \quad A = \frac{\ddot{s}}{s} - \left(1 - \frac{1}{n}\right) \frac{\dot{s}^2}{s^2} - a_1 \frac{\dot{s}}{s} + \frac{a_2}{s} - b_2.$$

Two equations corresponding to one another by the transformation $yu = 1$ are not considered as distinct; therefore, we have only to consider four classes of equations represented symbolically by $(P_2; Z_2)$, $(P_1; Z_2)$, $(P_2; Z_1)$, $(P_1; Z_1)$.

We consider each class separately and summarize the various stability conditions already obtained above (see § 17) with

tables where the corresponding values of p are also indicated.

P2;Z2. For P2 , one has $a = b = 0$ and for Z2 , $a_2 = b_4 = 0$.
The various cases to be considered are given in the following table :

n	P2	p	Z2	p
n	$b_1 = 0$	-	$b_3 = 0$	-
2	$b_1 \neq 0$	3	$b_3 = 0$	-
4	$b_1 \neq 0$	2	$b_3 \neq 0$	0
4	$b_1 = 0$	-	$b_3 \neq 0$	0

Thus , the stable equations of class (P2,Z2) are of the form

$$E.1 \quad \ddot{y} - \left(1 - \frac{1}{n} \right) \frac{\dot{y}^2}{y} = F(x,y) \quad ,$$

$$E.2 \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = b_1 y^2 + F(x,y) \quad ,$$

$$E.3 \quad \ddot{y} - \frac{3}{4} \frac{\dot{y}^2}{y} = b_1 y^2 + b_3 + F(x,y) \quad ,$$

$$E.4. \quad \ddot{y} - \frac{3}{4} \frac{\dot{y}^2}{y} = b_3 + F(x,y) \quad ,$$

where

$$F(x,y) = a_1 \dot{y} + b_2 y \quad .$$

P1;Z2. Because of the double parametric zeros , one has $a_2 = b_4 = 0$; in addition , we have also to consider , for $n \neq 0$, $b_3 = 0$ and for $n = 4$, $b_3 \neq 0$.

The various cases to be considered are given in the following table :

n	P1	p	Z2	p
n	$na^2 + b(n+2)^2 = 0$	$0, -(n+2)$	$b_3 = 0$	-
2	$a^2 = 2b$	1, 5	$b_3 = 0$	-
2	$a = 0, b \neq 0$	2, 2	$b_3 = 0$	-
3	$3a^2 = 2b$	1, 3	$b_3 = 0$	-
5	$5a^2 = b$	1, 2	$b_3 = 0$	-
4	$a^2 + 9b = 0$	0, -6	$b_3 \neq 0$	0

Thus, the stable equations of class (P1;Z2) are of the form

$$\text{E.5.} \quad \ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = ayy' - \frac{n a^2}{(n+2)^2} y^3 + F(x, y) \quad ,$$

$$\text{E.6.} \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = ayy' + \frac{a^2}{2} y^3 + F(x, y) \quad ,$$

$$\text{E.7.} \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = b y^3 + F(x, y) \quad ,$$

$$\text{E.8.} \quad \ddot{y} - \frac{2}{3} \frac{\dot{y}^2}{y} = ayy' + \frac{3}{2} a^2 y^3 + F(x, y) \quad ,$$

$$\text{E.9.} \quad \ddot{y} - \frac{4}{5} \frac{\dot{y}^2}{y} = ayy' + 5 a^2 y^3 + F(x, y) \quad ,$$

$$\text{E.10.} \quad \ddot{y} - \frac{3}{4} \frac{\dot{y}^2}{y} = ayy' - \frac{a^2}{9} y^3 + b_3 + F(x, y) \quad ,$$

where

$$F(x, y) = a_1 \dot{y} + b_1 y^2 + b_2 y \quad .$$

P2;Z1. We assume $n > 0$. Because of the double parametric poles, one has $a = b = 0$. The various cases to be considered are given in the following table

n	P2	p	Z1	p
$n \neq 2$	$b_1 = 0$	-	$na_2^2 + (n-2)^2 b_4 = 0$	$0, n-2$
$n = 2$	$b_1 = 0$	-	$a_2 = 0, b_4 \neq 0$	$0, 0$
4	$b_1 \neq 0$	2	$a_2^2 + b_4 = 0$	$0, 2$
2	$b_1 \neq 0$	3	$a_2 = 0, b_4 \neq 0$	$0, 0$

Thus, the stable equations of class (P2;Z1) are of the form

$$\text{E.11.} \quad \ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = a_2 \frac{\dot{y}}{y} - \frac{n a_2^2}{(n-2)^2 y} + F(x, y), n \neq 2,$$

$$\text{E.12.} \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = \frac{b_4}{y} + F(x, y),$$

$$\text{E.13.} \quad \ddot{y} - \frac{3}{4} \frac{\dot{y}^2}{y} = a_2 \frac{\dot{y}}{y} - \frac{a_2^2}{y} + b_1 y^2 + F(x, y),$$

$$\text{E.14.} \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = b_1 y^2 + \frac{b_4}{y} + F(x, y),$$

where

$$F(x, y) = a_1 \dot{y} + b_2 y + b_3.$$

P1;Z1. The various cases to be considered are given in the following table :

n	P1	p	Z1	p
$n \neq 2$	$na^2 + (n+2)^2 b = 0$	$0, -(n+2)$	$na_2^2 + (n-2)^2 b_4 = 0$	$0, n-2$
2	$a^2 + 8b = 0$	$0, -4$	$a_2 = 0, b_4 \neq 0$	$0, 0$
2	$a^2 = 2b$	$1, 5$	$a_2 = 0, b_4 \neq 0$	$0, 0$
2	$a = 0, b \neq 0$	$2, 2$	$a_2 = 0, b_4 \neq 0$	$0, 0$
3	$3a^2 = 2b$	$1, 3$	$3a_2^2 + b_4 = 0$	$0, 1$
5	$5a^2 = b$	$1, 2$	$5a_2^2 + 9b_4 = 0$	$0, 3$

Thus, the stable equations of class (P1,Z1) are of the form

$$E.15. \quad \ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = \left(ay + \frac{a_2}{y}\right) \dot{y} - \frac{na^2}{(n+2)^2} y^3 - \frac{na_2^2}{(n-2)^2 y} + F(x, y),$$

$$E.16. \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = ayy - \frac{a^2}{8} y^3 + \frac{b_4}{y} + F(x, y),$$

$$E.17. \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = ayy + \frac{a^2}{2} y^3 + \frac{b_4}{y} + F(x, y),$$

$$E.18. \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = by^3 + \frac{b_4}{y} + F(x, y),$$

$$E.19. \quad \ddot{y} - \frac{2}{3} \frac{\dot{y}^2}{y} = \left(ay + \frac{a_2}{y}\right) \dot{y} + \frac{3}{2} a^2 y^3 - \frac{3a_2^2}{y} + F(x, y),$$

$$E.20. \quad \ddot{y} - \frac{4}{5} \frac{\dot{y}^2}{y} = \left(ay + \frac{a_2}{y}\right) \dot{y} + 5a^2 y^3 - \frac{5a_2^2}{9y} + F(x, y),$$

where

$$F(x, y) = a_1 \dot{y} + b_1 y^2 + b_2 y + b_3.$$

Comparing the values of p for P1 in the tables (P1,Z2) and (P1,Z1), one is led to study together the equations E.6 and E.17, E.7 and E.18, E.8 and E.19, E.9 and E.20.

Theorem IV gives additional conditions for stability .
We always begin by considering the smallest values of p .

38. Equation E.1. Using a transformation (λ, φ) , we may assume $a_1 = b_2 = 0$. Then , equation E.1 is

$$(38.1) \quad \ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = 0 \quad ;$$

its general solution is given by $y = (Kx + H)^n$ and is stable .
On setting $y = z^n$, equation (38.1) reduces to $\ddot{z} = 0$.

39. Equation E.2. For P2 , one has $p = 3$, $b_1 s = 4$; by a transformation (λ, φ) , one may assume $a_1 = 0$, $b_4 = 4$ or $s=1$.

Now , set $y = z^{-2}$, $\dot{z} = 1 + u z$ and determine u by

$$z \dot{u} = 3 u - \frac{b_2}{2} z + z u^2 .$$

To obtain the condition for stability , set $u = \beta z + \gamma z^2 + z^3 v$ and determine β, γ by

$$4\beta = b_2 , \quad \dot{\beta} = \gamma , \quad \dot{\gamma} = 0 \quad ;$$

therefore , γ is a constant and $b_2 = Kx + H$.

The stable equation E.2 is thus of the form

$$(39.1) \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = 4 y^2 + (Kx + H) y .$$

According to the values of K and H , one finds one or the other of the three following canonical equations .

a. $K = H = 0$. The equation

$$(39.2) \quad \ddot{y} = \frac{1}{2} \frac{\dot{y}^2}{y} + 4 y^2$$

may be integrated by the method of variation of parameters.

One finds

$$\dot{y}^2 = y (4y^2 + K_1) \quad ;$$

thus , $y(x)$ is an elliptic function and is stable .

One may also set $y = z^2$ and determine z by $\ddot{z} = 2 z^3$.

b. $K = 0$, $H \neq 0$. A linear transformation on x reduces (39.1) to

$$(39.3) \quad \ddot{y} = \frac{1}{2} \frac{\dot{y}^2}{y} + 4 y^2 + 2 y .$$

This equation may be integrated by the method of variation of parameters . One finds

$$\dot{y}^2 = 4 y (y^2 + y + K_1) \quad ;$$

thus , $y(x)$ is an elliptic function and is stable .

One may also set $y = z^2$ and determine z by $\ddot{z} = 2 z^3 + z$.

c. $K.H \neq 0$. A trivial change of variable brings equation (39.1) to

$$(39.4) \quad \ddot{y} = \frac{1}{2} \frac{\dot{y}^2}{y} + 4 y^2 + 2 x y$$

which on setting $y = z^2$ becomes

$$\ddot{z} = 2 z^3 + x z \quad ,$$

i.e. equation II of Table I .

40. Equation E.3. For P2, one has $p = 2$, $b_1 s = 4$. By a transformation (λ, ϕ) , one may assume $a_1 = 0$, $b_1 = 4$ and $s = 1$. Now set $y = z^{-2}$, $\dot{z} = 1 + u z$ and determine u by

$$z \dot{u} = 2u - \frac{1}{2} (b_2 z + b_3 z^3) + \frac{z}{2} u^2.$$

We obtain the condition for stability by setting $u = \beta z + z^2 v$, where

$$2\beta = b_2, \quad \dot{\beta} = 0;$$

therefore, $b_2 = 2K$.

For Z2, we have according to the table of equivalence, $p = 0$, $b_3 s + 1 = 0$. Set $y = \frac{z^2}{s}$, $\dot{z} = 1 + u z$ and determine u by

$$z \dot{u} = -\frac{\dot{s}}{2s} + \left(\frac{\ddot{s}}{2s} - \frac{5}{8} \frac{\dot{s}^2}{s^2} z + K \right) z + \frac{3}{2s} z^2 + \frac{\dot{s}}{2s} uz - \frac{z}{2} u^2.$$

The condition for stability is $\dot{s} = 0$ or $b_3 = H$.

The stable equation E.3 is thus of the form

$$(40.1) \quad \ddot{y} = \frac{3}{4} \frac{\dot{y}^2}{y} + 3y^2 + 2Ky + H.$$

The general solution of (40.1) may be determined by the method of variation of parameters. One finds

$$\dot{y}^2 = 4 \left[y^3 + Ky^2 - Hy + K_1 y^{3/2} \right]$$

or on setting $y = z^2$,

$$\dot{z}^2 = z^4 + Kz^2 - H + K_1 z;$$

thus, $y(x)$ is an elliptic function and is stable.

41. Equation E.4. By a transformation (λ, φ) , we may assume $a_1 = b_2 = 0$. There is no condition for stability for P2. For Z2, one has $p = 0$, $b_3 s + 1 = 0$; the condition for stability is $\dot{s} = 0$ or $b_3 = K$.

The stable equation E.4 is of the form

$$(41.1) \quad \ddot{y} = \frac{3}{4} \frac{\dot{y}^2}{y} + K.$$

If $K = 0$, we have a particular case of (38.1); $y(x)$ is stable.

If $K \neq 0$, set $y = -K u$ and find the canonical equation

$$\ddot{u} = \frac{3}{4} \frac{\dot{u}^2}{u} - 1.$$

The general solution may be determined by the method of variation of parameters or by setting $u = z^2$. One finds $2z\ddot{z} = \dot{z}^2 - 1$ and upon differentiation $\ddot{z} = 0$. Therefore, $y(x)$ is stable.

42. Equation E.5. By a transformation (λ, φ) , we may assume $a_1 = b_2 = 0$. Then, the equation E.5 is

$$\ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = a y \dot{y} - \frac{n a^2}{(n+2)^2} y^3 + b_1 y^2.$$

The only condition for stability is produced by P1, $p = 0$, $n a s + n + 2 = 0$ and according to (37.4) is

$$b_1 s + a \dot{s} + \frac{2s}{ns} = 0$$

or

$$b_1 = \frac{n}{n+2} \dot{a}.$$

The stable equation E.5 is of the form

$$(42.1) \quad \ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = a y \dot{y} - \frac{n a^2}{(n+2)^2} y^3 + \frac{n}{n+2} \dot{a} y^2 ,$$

where a is an arbitrary analytic function of x .

To integrate (42.1) , note that

$$(42.2) \quad \frac{\dot{y}}{y} - \frac{n a y}{n+2} = n v$$

is regular at the parametric poles . Then v is given by

$$\dot{v} + v^2 = 0 \text{ or } v = \frac{1}{x+H} ; \text{ therefore , } y \text{ is determined by the}$$

Bernouilli equation

$$\dot{y} = \frac{n}{x+H} y + \frac{n a y^2}{n+2}$$

or on setting $y = \frac{1}{w}$, by the linear differential equation of

the first order

$$\dot{w} + \frac{nw}{x+H} + \frac{n a}{n+2} = 0 .$$

On setting $(x+H)^n w = W$, one obtains

$$\dot{W} + \frac{na}{n+2} (x+H)^n = 0 ;$$

thus , $y(x)$ is stable .

More generally , suppose that a_1 and b_2 are not identically zero . Then , the condition of stability corresponding to P_1 , $p = 0$ is

$$\dot{a} - a_1 a - \frac{n+2}{n} b_1 = 0$$

and gives b_1 .

The equation E.5 becomes

$$(42.3) \quad \ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = a_1 \dot{y} - \frac{n a^2}{(n+2)^2} y^3 + \frac{n}{n+2} (\dot{a} - a_1 a) y^2 + b_2 y.$$

To integrate this equation, determine again y by (42.2) and set

$v = \frac{\dot{w}}{w}$; w is given by

$$\ddot{w} - a_1 \dot{w} - \frac{b_2}{n} w = 0,$$

and y is a solution of the Bernoulli equation

$$\dot{y} = n \frac{\dot{w}}{w} y + \frac{n a}{n+2} y^2.$$

On setting $y = \frac{w^n}{u}$, one finds

$$\dot{u} + \frac{n a}{n+2} w^n = 0;$$

therefore, u and y are stable.

43. Equation E.10. Both P1 and Z2 give a condition for stability. For P1, we observe that the condition $a^2 + 9b = 0$ is identical to the condition given for equation E.5 where $n = 4$. Then, $p = 0$, $2as + 3 = 0$ and by a transformation (λ, φ) , one may assume $a_1 = 0$, $a = -\frac{3}{2}$ and $s = 1$. The condition of stability is $b_1 = 0$.

For Z2, on using the table of equivalence, one has $p = 0$, $1 + b_3 s = 0$; the condition for stability is $\dot{s} = 0$ or $b_3 = K \neq 0$.

Thus, the stable equation E.10 is of the form

$$(43.1) \quad \ddot{y} - \frac{3}{4} \frac{\dot{y}^2}{y} = -\frac{3}{2} y \dot{y} - \frac{1}{4} y^3 + b_2 y + K,$$

where b_2 is an arbitrary analytic function of x .

To integrate (43.1) , observe that it is equivalent to the differential system

$$\begin{aligned}\dot{y} + y^2 &= 2 v y \quad , \\ y &= \frac{K}{2 \dot{v} + v^2 - b_2} \quad .\end{aligned}$$

On eliminating y between these relations (consider $\frac{\dot{y}}{y}$) , one obtains

$$\ddot{v} + 3 v \dot{v} + v^3 - b_2 v - \frac{1}{2} (\dot{b}_2 + K) = 0 \quad .$$

This is an equation of type $A(x,y) = 0$ and class P1 , $p = 0$ [see § 23 , eq. 20.4] ; on setting $v = \frac{\dot{w}}{w}$, it reduces to

the linear equation

$$\ddot{w} - b_2 \dot{w} - \frac{1}{2} (\dot{b}_2 + K) w = 0 \quad .$$

Therefore , y is stable .

Remark . We may also choose (λ, φ) such that $a_1 = b_1$, $a = -\frac{3}{2}$, [Gambier , p. 28 , eq. 2] ; then , one finds the equation

$$\ddot{y} = \frac{3}{4} \frac{\dot{y}^2}{y} - \frac{3}{2} y \dot{y} - \frac{y^3}{4} + \frac{\dot{q}}{2q} (\dot{y} + y^2) + ry + q \quad ,$$

where q and r are arbitrary analytic functions of x .

This equation is equivalent to the differential system

$$\begin{aligned}\dot{y} + y^2 &= 2 v y \quad , \\ y \left[2 \dot{v} + v^2 - \frac{\dot{q}}{q} v - r \right] &= q \quad .\end{aligned}$$

On eliminating y between these relations, one finds

$$\ddot{v} + 3\dot{v}\dot{v} - \frac{3}{2} \frac{\dot{q}}{q} \dot{v} + v^3 - \frac{3\dot{q}}{2q} v^2 - \left(\frac{\ddot{q}}{2q} - \frac{\dot{q}^2}{q^2} + r \right) v - \frac{1}{2} \left(\dot{r} + \dot{q} - \frac{\dot{q}r}{q} \right) = 0.$$

On setting $v = \frac{\dot{w}}{w}$, this equation of type $A(x, y) = 0$ and class

P1, $p = 0$ reduces to the linear equation

$$\ddot{w} - \frac{3}{2} \frac{\dot{q}}{q} \dot{w} + \left(\frac{\ddot{q}}{2q} - \frac{\dot{q}^2}{q^2} + r \right) \dot{w} + \frac{1}{2} \left(\dot{r} + \dot{q} - \frac{\dot{q}r}{q} \right) w = 0.$$

44. Equation E.11. Suppose $n \neq 2$ and consider the equation

$$\ddot{y} - \left(1 - \frac{1}{n} \right) \frac{\dot{y}^2}{y} =$$

(44.1)

$$= a_2 \frac{\dot{y}}{y} - \frac{na_2^2}{(n-2)^2 y} + a_1 \dot{y} + b_2 y + b_3.$$

For P2, there is no condition for stability.

For Z1, one has

$$n a_2 s = -(n-2), \quad p = 0;$$

$$n a_2 s = (n-2)(n-1), \quad p = n-2.$$

A transformation (λ, φ) may be chosen such that $a_1 = 0$,
 $a_2 = -\frac{n-2}{2}$. The condition for stability for Z1, $p = 0$,

($s = 1$) is then $b_3 = 0$.

Equation E.11 is then

$$(44.2) \quad \ddot{y} - \left(1 - \frac{1}{n} \right) \frac{\dot{y}^2}{y} = -\frac{n-2}{n} \frac{\dot{y}}{y} - \frac{1}{ny} + by,$$

where b is determined by the condition for stability for Z1,

$p = n-2$, $s = 1-n$. To obtain this condition (call it

condition A) , set $u = P_{n-3}(z) + z^{n-2}U$ and write $n-2$ linear equations .

For instance , suppose $n = 4$; then

$$z \dot{u} = 2u + bz - \frac{z}{2} u^2 .$$

On setting $u = \beta z + z^2 U$, one finds $\beta + b = P$, $\dot{\beta} = 0$ and hence the condition of stability $b = K$.

To integrate (44.2) , observe that $\frac{\dot{y}}{y} - \frac{1}{y}$ is regular for the parametric zeros $Z1$, $p = 0$ and set

$$\dot{y} = 1 + n v y \quad ;$$

then , v is given by the Riccati equation

$$\dot{v} + v^2 = \frac{1}{n} b$$

or if $v = \frac{\dot{w}}{w}$ by $\ddot{w} = \frac{b}{n} w$. Accordingly , y is given by

$$y = w^n \left[K + \int \frac{dx}{w^n} \right] .$$

If $w(x)$ has a simple parametric zero ($x = x_0$) , then $y(x) = (x - x_0) s(x)$; for $y(x)$ to be stable , b must satisfy the condition of stability A .

45. Equation E.12. Consider the equation

$$(45.1) \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = \frac{b_4}{y} + a_1 \dot{y} + b_2 y + b_3 .$$

For $P2$, there is no condition for stability .

For $Z1$, one has according to the table of equivalence ,

$$2b_4 s^2 + 1 = 0 , p = 0 , 0 .$$

A transformation (λ, φ) may be chosen such that $b_2 = 0$,
 $b_4 = -\frac{1}{2}$; then the condition for stability for Z1 is

$a_1 + b_3 s = 0$ or , because $s = \pm 1$, $a_1 = b_3 = 0$.

The equation E.12 is then

$$(45.2) \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = -\frac{1}{2y}$$

and may be integrated by the method of variation of parameters.

One finds $\dot{y}^2 = uy$, $u = \frac{1}{y} + 2K$ and hence

$$(45.3) \quad \dot{y}^2 = 1 + 2Ky .$$

On differentiating (45.3) , one obtains $\ddot{y} = K$ and

$$(45.4) \quad y(x) = \frac{K}{2} x^2 + K_1 x + K_2 ;$$

therefore , $y(x)$ is stable .

On substituting $y(x)$ given by (44.4) into (45.3) , one finds $K_1^2 = 1 + 2KK_2$.

46. Equation E.13. For P2 , one has $p = 2$, $b_1 s = 3$; a transformation (λ, φ) may be chosen such that $a_1 = 0$, $b_3 = 3$ and $s = 1$. The equation giving u is then

$$z\dot{u} = 2u - \frac{1}{2} b_2 z + a_2 z^2 - \frac{b_3}{2} z^3 + \frac{a_2^2}{2} z^5 + a_2 z^3 u + \frac{z}{2} u^2 .$$

To find the condition for stability , set $u = \beta z + z^2 v$ and obtain

$$2\beta = b_2 , \quad \dot{\beta} = a_2 .$$

For convenience , set $b_2 = 12q$, $\dot{a}_2 = 6\dot{q}$; then , equation

E.13 becomes

$$(46.1) \quad \ddot{y} - \frac{3}{4} \frac{\dot{y}^2}{y} = 6 \dot{q} \frac{\dot{y}}{y} + 3 y^2 + 12 q y + b_3 - \frac{36 \dot{q}^2}{y} .$$

For Z1 , one has $p = 0$, $12 \dot{q} s + 1 = 0$; $p = 2$, $12 \dot{q} s = 3$.

From the table of equivalence , one finds that $n = -4$ and that for $p = 0$, the condition for stability is

$$b_3 s - 6 \dot{q} \dot{s} + \frac{\dot{s}}{2s} = 0 ;$$

because $s = -\frac{1}{12 \dot{q}}$, one has $b_3 = -12 \ddot{q}$.

The condition for stability corresponding to Z1 , $p = 2$, determines q . Indeed , because $4 \dot{q} s = 1$, one deduces from (37.4) , the equation

$$(46.2) \quad z \dot{u} = 2u - 2 \frac{\ddot{q}}{\dot{q}} + Az + 12 \dot{q} z^2 - \frac{\ddot{q}}{2\dot{q}} z u - \frac{zu^2}{4}$$

with

$$A = -\frac{\ddot{\dot{q}}}{\dot{q}} + \frac{3}{4} \frac{\ddot{q}^2}{\dot{q}^2} + 12 q .$$

To find the condition for stability , set $u = \alpha + \beta z + z^2 v$ into (46.2) ; one obtains

$$\alpha = \frac{\ddot{q}}{\dot{q}} , \quad \beta = 2 \frac{\ddot{\dot{q}}}{\dot{q}} - \frac{\ddot{q}^2}{\dot{q}^2} - 12 q , \quad \dot{\beta} = 12 \dot{q} - 2\beta \frac{\ddot{q}}{\dot{q}}$$

so that , on eliminating β ,

$$q^{(4)} = 12 \dot{q}^2 + 12 q \ddot{q}$$

or

$$(46.3) \quad \ddot{q} = 6 q^2 + K x + H .$$

Therefore , the stable equation of this class is of the form

$$(46.4) \quad \ddot{y} = \frac{3}{4-y} \frac{\dot{y}^2}{y} + 6 \dot{q} \frac{\dot{y}}{y} + 3 y^2 + 12 q y - 12 \ddot{q} - \frac{36 \dot{q}^2}{y} ,$$

where q is a solution of (46.3) .

To integrate (46.4) , we use a method which gives also the values of q . Observe that $\dot{y} + 12 \dot{q}$ is regular for the parametric zeros corresponding to $p = 0$. Therefore , the equation (46.4) is equivalent to the differential system

$$\begin{aligned} \dot{y} + 12 \dot{q} &= - 2 v y , \\ 3 y &= - 2 \dot{v} + v^2 - 12 q \end{aligned}$$

which shows that v is stable if y is stable and conversely .

On eliminating y between these relations , it is readily seen that v is determined by the equation

$$\ddot{v} = - v \dot{v} + v^3 - 12 q v + 12 \dot{q}$$

which is of type $A(x,y) \equiv 0$ and class $P1$, $p = 2$ and 5 [see (25.5)] .

For this equation to be stable , it is necessary that q satisfies (46.3) ; the general solution of the corresponding equation is then given by

$$v = \frac{\dot{V} - \dot{q}}{V - q} ,$$

where V is a solution of (46.3) distinct from q . Therefore , y is stable .

According to the values of K and H , one finds one or the other of the three following canonical equations .

a. $K = H = 0$. Then (46.3) is $\ddot{q} = 6 q^2$ or $\dot{q}^2 = 4 q^3 + K_1$.

When $K_1 = 0$, one has $q = (x + K_2)^{-2}$; a trivial change of variable ($x + K_2 \rightarrow x$) brings equation (46.4) to

$$(46.5) \quad \ddot{y} = \frac{3}{4} \frac{\dot{y}^2}{y} + 3y^2 - \frac{12}{x} \frac{\dot{y}}{y} + \frac{12y}{x} - \frac{72}{x^4} - \frac{144}{x^6 y}.$$

When $K_1 \neq 0$, one has $q = \wp(x; 0, K_1)$ where \wp is the well known elliptic function of Weierstrass; equation (46.4) is then

$$(46.6) \quad \ddot{y} = \frac{3}{4} \frac{\dot{y}^2}{y} + 3y^2 + 6 \wp \frac{\dot{y}}{y} + 12 \wp y - 12 \ddot{\wp} - \frac{36 \dot{\wp}^2}{y}.$$

b. $K = 0$, $H \neq 0$. Then $\ddot{q} = 6q^2 + H$, $\dot{q}^2 = 4q^3 + 2Hq + H_1$; therefore, $q = \wp(x; -2H, H_1)$ and (46.4) has the same form as (46.6).

c. $K \neq 0$. On using a trivial change of variable, one may assume $K = 1$, $H = 0$ so as to reduce (46.3) to $\ddot{q} = 6q^2 + x$.

47. Equation E.14. For P2, one has $p = 3$, $b_1 s = 4$ and for Z1, $p = 0$, $2b_4 s^2 + 1 = 0$. By a transformation (λ, φ) , one may assume $b_1 = 4$ and $b_4 = -\frac{1}{2}$.

First, we consider Z1. The condition for stability is $a_1 + b_3 s = 0$, $s = \pm 1$ so that $a_1 = 0$, $b_3 = 0$.

For P2, one has $s = 1$ and

$$z\dot{u} = 3u - \frac{1}{2} b_2 z + \frac{1}{4} z^5 + z u^2.$$

To obtain the condition for stability, set $u = \beta z + \gamma z^2 + z^3 v$ and find

$$4\beta = b_2, \quad \dot{\beta} = \gamma, \quad \dot{\gamma} = 0$$

so that $b_2 = Kx + H$.

The stable equation E.14 is of the form

$$(47.1) \quad \ddot{y} = \frac{1}{2} \frac{\dot{y}^2}{y} + 4y^2 + (Kx + H)y - \frac{1}{2y}.$$

According to the values of K , we have the following canonical equations.

a. $K = 0$. One has

$$(47.2) \quad \ddot{y} = \frac{\dot{y}^2}{2y} - \frac{1}{2y} + 4y^2 + 2Hy;$$

the method of variation of parameters gives

$$\dot{y}^2 = 4y^3 + 4Hy^2 + 2H_1y + 1.$$

Therefore, $y(x)$ is an elliptic function and is stable.

b. $K \neq 0$. A trivial change of variables ($Kx+H \rightarrow x, K^2y \rightarrow y$) reduces (47.1) to

$$(47.3) \quad \ddot{y} = \frac{\dot{y}^2}{2y} - \frac{1}{2y} + 4Hy^2 - xy, \quad H \neq 0.$$

To integrate (47.3), observe that $\frac{\dot{y}}{y} + \frac{1}{y} = 2v$ is regular for the parametric zeros corresponding to $s = -1$. The equation (47.3) is equivalent to the differential system

$$\begin{cases} \dot{y} = 2vy - 1, \\ 4Hy = 2\dot{v} + 2v^2 + x. \end{cases}$$

On eliminating y between the relations, one finds

$$\ddot{v} = 2v^3 + xv - 2H - \frac{1}{2}$$

so that v is a solution of a Painlevé equation II (see Table I).

Therefore, $y(x)$ is stable.

48. Equation E.15. Consider the equation

$$(48.1) \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = a y \dot{y} - \frac{a^2}{8} y^3 + \frac{b_4}{y} + a_1 \dot{y} + b_1 y^2 + b_2 y + b_3.$$

For P1, one has $p = 0$, as $+2 = 0$; there is no condition for stability for P1, $p = -4$.

For Z1, one has $2b_4 s^2 + 1 = 0$, $p = 0, 0$.

A transformation (λ, φ) may be chosen such that $a = -2$, $b_4 = -\frac{1}{2}$. The conditions for stability are

for P1, $a_1 = b_1$;

for Z1, $a_1 + b_3 s = 0$, $s = \pm 1$ and hence $a_1 = b_3 = 0$.

Therefore, the equation (48.1) can be written in the form

$$(48.2) \quad \ddot{y} - \frac{\dot{y}^2}{2y} = -2y \dot{y} - \frac{y^3}{2} - \frac{1}{2y} + f y,$$

where f is an arbitrary analytic function of x .

To integrate equation (48.2), observe that $\frac{\dot{y}}{y} + y$ is regular for the parametric poles P1, $p = 0$ and that $\frac{\dot{y} - 1}{y}$ is

regular for the parametric zeros Z1. Accordingly, set

$$(48.3) \quad \dot{y} = -y^2 + 1 + 2v y;$$

substitution in (48.2) gives

$$(48.4) \quad \dot{v} + v^2 = \frac{1}{2} (f - 1)$$

and $y(x)$ is determined by the two Riccati equations (48.3-4).

To prove that $y(x)$ is stable, set $y = \frac{\dot{w}}{w}$; then from

(48.3), one obtains $2v \dot{w} = \ddot{w} - w$ and (48.4) becomes

$$2 \dot{w} w = \ddot{w}^2 - w^2 + 2f \dot{w}^2,$$

whence , upon differentiation ,

$$w^{(4)} - 2 f \ddot{w} - \dot{f} \dot{w} + w = 0 .$$

Thus , $w(x)$ is stable and also $y(x)$.

More generally , if $a \neq -2$ and $b_4 \neq -\frac{1}{2}$, one finds

the conditions for stability

$$(48.5) \quad \begin{cases} \text{for P1} : 2 b_1 = \dot{a} - a a_1 ; \\ \text{for Z1} : b_3 = 0 \text{ and } \dot{b}_4 = 2 a_1 b_4 . \end{cases}$$

On setting $b_4 = r$, the stable equation E.15 becomes

$$(48.6) \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} =$$

$$= a y \dot{y} + \frac{\dot{r}}{2 r} \dot{y} - \frac{a^2}{8} y^3 + \frac{1}{2} \left(\dot{a} - \frac{a \dot{r}}{2 r} \right) y^2 + b_2 y + \frac{r}{y} .$$

49. Equation E.14. Suppose $n \neq 2$ and consider the equation

$$(49.1) \quad \ddot{y} - \left(1 - \frac{1}{n} \right) \frac{\dot{y}^2}{y} = \left(a y + \frac{a_2}{y} \right) \dot{y} - \frac{n a^2}{(n+2)^2} y^3 -$$

$$- \frac{n a_2^2}{(n-2)^2 y} + a_1 \dot{y} + b_1 y^2 + b_2 y + b_3 .$$

Note that E.14 reduces to E.11 when $a = b_1 = 0$. One has for P1 :

$$n a s = - (n+2) , \quad p = 0 ;$$

$$n a s = - (n+2) (n-1) , \quad p = - (n+2) ;$$

for Z1 :

$$n a_2 s = - (n-2) , \quad p = 0 ;$$

$$n a_2 s = (n-2) (n-1) , \quad p = n-2 .$$

A transformation (λ, φ) may be chosen such that $a = -\frac{n+2}{n}$

and $a_2 = -\frac{n-2}{n}$. Then, equation (49.1) becomes

$$\ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = -\frac{n+2}{n} y\dot{y} - \frac{n-2}{n} \frac{\dot{y}}{y} -$$

(49.2)

$$-\frac{y^3}{n} - \frac{1}{ny} + a_1\dot{y} + b_1y^2 + b_2y + b_3.$$

The conditions for stability are

for P1, $p = 0$, $s = 1$: $a_1 = b_1$;

for Z1, $p = 0$, $s = 1$: $a_1 + b_3 = 0$;

for P1, $p = -(n+2)$, there is no condition for stability.

Now, set $a_1 = b_1 = -b_3 = b$ and $b_2 = f$; then equation (49.2) can be rewritten

$$\ddot{y} - \left(1 - \frac{1}{n}\right) \frac{\dot{y}^2}{y} = -\frac{n+2}{n} y\dot{y} - \frac{n-2}{n} \frac{\dot{y}}{y} -$$

(49.3)

$$-\frac{y^3}{n} - \frac{1}{ny} + b(\dot{y} + y^2 - 1) + f y.$$

It remains to consider the condition for stability for Z1, $p = n-2$, $s = 1-n$; this condition (call it condition A) gives a relation between b and f and their derivatives. To find this condition A, set as usual

$$u = P_{n-3}(z) + z^{n-2} U$$

and apply our basic theorem IV.

For instance, consider $n = 2$ and $n = 3$.

For $n = 2$, one has

$$\ddot{y} - \frac{1}{2} \frac{\dot{y}}{y} = -2y\dot{y} - \frac{y^3}{2} - \frac{1}{2y} + b(\dot{y} + y^2 - 1) + fy$$

and condition A is $b = 0$; one finds equation E.15 (see 48.2).

For $n = 3$, one has

$$\ddot{y} - \frac{2}{3} \frac{\dot{y}}{y} = -\frac{5}{3} y\dot{y} - \frac{1}{3} \frac{\dot{y}}{y} - \frac{y^3}{3} - \frac{1}{3y} + b(\dot{y} + y^2 - 1) + fy$$

and

$$z\dot{u} = u + 3b + \left(\frac{5}{6} + f\right)z - \frac{bz^2}{2} - \frac{z^3}{12} + bzu + \frac{5}{6} z^2 u - \frac{z}{3} u^2;$$

the condition for stability is

$$(49.4) \quad 3\dot{b} + f + \frac{5}{6} - 6b^2 = 0$$

and determines f .

Suppose that the condition A for stability has been determined.

To integrate (49.3) , set

$$\dot{y} = -y^2 + 1 + nvy$$

so that

$$\dot{v} + v^2 = bv + \frac{1}{n} \left(f - \frac{2}{n} \right) .$$

Now , set $y = \frac{\dot{u}}{u}$, $v = \frac{\dot{w}}{w}$; w and u are given by

$$(49.5) \quad \ddot{w} - b\dot{w} - \frac{1}{n} \left(f - \frac{2}{n} \right) w = 0 ,$$

$$(49.6) \quad \ddot{u} - n \frac{\dot{w}}{w} \dot{u} - u = 0 .$$

The singular points of w are fixed . Let $x = x_0$ be a simple parametric zero of $w(x)$; then $x = x_0$ is a regular singular point (in the sense of Fuchs) of (49.6) . Note that the roots

of the corresponding indicial equation are 0 and $n+1$; therefore , $u(x)$ may have a logarithmic term except if a certain condition (call it condition B) is satisfied . In this case , equation (49.6) has two regular solutions

$$u_1(x) = (x - x_0)^{n+1} [1 + \dots] ,$$

$$u_2(x) = 1 + \frac{(x - x_0)^2}{2(1 - n)} + \dots$$

to which correspond respectively

$$y_1(x) = \frac{n+1}{x - x_0} + \dots ,$$

$$y_2(x) = \frac{x - x_0}{1 - n} + \dots .$$

Therefore , conditions A and B coincide and $y(x)$ is stable .

50. We now recall the method used by B. Gambier to determine the stable equations E.14. Set

$$(50.1) \quad \dot{y} = -y^2 - \frac{1}{n-1} + n v y$$

and substitute into equation (49.3) ; then

$$(50.2) \quad y \left[\dot{v} + v^2 - bv - \frac{f}{n} - \frac{n^2 - 2n + 2}{n^2(n-1)} \right] = -\frac{b}{n-1} - \frac{n-2}{n-1} v .$$

To abbreviate , set

$$\alpha = -\frac{nb}{n-2} , \quad v = z + \frac{\alpha}{n} ,$$

$$\beta = -\frac{\dot{b}}{n-2} + \frac{n-1}{(n-2)^2} b^2 - \frac{f}{n} - \frac{n^2 - 2n + 2}{n^2(n-1)} ,$$

$$P(z) = z^2 + \alpha z + \beta$$

so that (50.1) and (50.2) become respectively

$$(50.3) \quad \dot{y} = -y^2 - \frac{1}{n-1} + \alpha y + n z y, \quad ,$$

$$(50.4) \quad y \left[\dot{z} + P(z) \right] = - \frac{n-2}{n-1} z.$$

Hence, if $y(x)$ is stable, $v(x)$ is also stable and vice versa.

On eliminating y between (50.3) and (50.4), one finds

$$(50.5) \quad \ddot{z} = \left(1 - \frac{1}{n-2} \right) \frac{\dot{z}^2}{z} - \frac{n(n-1)}{n-2} z \dot{z} - \frac{n\alpha}{n-2} \dot{z} + \frac{n-4}{n-2} \beta \frac{\dot{z}}{z} - \frac{(n-1)^2}{n-2} z^3 - \frac{n(n-1)\alpha}{n-2} z^2 - B_2 z - B_3 - \frac{\beta^2}{n-2} \cdot \frac{1}{z},$$

where

$$B_2 = \dot{\alpha} + \frac{n-1}{n-2} \alpha^2 + \frac{n^2 - 2n + 2}{n-2} \beta - \frac{n-2}{n-1},$$

$$B_3 = \dot{\beta} + \frac{n}{n-2} \alpha \beta.$$

Equation (50.5) is an equation of the type E.14 with n replaced by $n-2$. Therefore, we have to consider two cases according as to whether n is even or odd.

a) $n = 2k$. Suppose $n = 4$. The equation (50.5) becomes

$$(50.6) \quad \ddot{z} = \frac{1}{2} \frac{\dot{z}^2}{z} - 6z\dot{z} - 2\alpha\dot{z} - \frac{9}{2} z^3 - 6\alpha z^2 - B_2 z - (\dot{\beta} + 2\alpha\beta) - \frac{\beta^2}{2z}.$$

If $\beta = 0$, this equation is stable and of the type E.5 for $n = 2$ [see 42.3].

If $\beta \neq 0$, equation (50.6) is an equation of the type E.15 [see 48.6] and is stable if the condition for stability $\dot{\beta} + 2\alpha\beta = 0$ is satisfied.

For $n = 2k > 4$, repeated applications of the method give equations of types E.5 or E.15 and hence k distinct classes of stable equations.

b) $n = 2k + 1$. Suppose $n = 3$. One has $\beta = 0$ [see (49.4)]. For $n = 2k + 1 > 3$ and $\beta = 0$, equation (50.5) is an equation of the type E.11 (see 42.3) and is stable. If $\beta \neq 0$, repeated applications of the method give again equations of the type E.11 and hence k distinct classes of stable equations.

51. Equations E.6 and E.17. Consider the equation

$$(51.1) \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = a_1 \dot{y} + \frac{a^2}{2} y^3 + b_1 y^2 + b_2 y + b_3 + \frac{b_4}{y}.$$

If $b_4 \neq 0$, we have equation E.17 and if $b_3 = b_4 = 0$, equation E.6.

Assume $b_4 \neq 0$. Then for Z1, one has $\dot{p} = 0$, $2b_4 s^2 + 1 = 0$. The condition for stability is $a_1 + b_3 s + \frac{s}{s} = 0$ so that

$$b_3 = 0, \quad 2a_1 = \frac{\dot{b}_4}{b_4}. \quad \text{For convenience, set } a_1 = q = \frac{\dot{r}}{r} \text{ and}$$

$b_4 = -72 H r^2$, where H is a constant. Note that $H = 0$ gives equation E.6.

Now, consider P1; one has $p = 1$, $a_s = -1$ and $p = 5$, $a_s = 3$. By a transformation (λ, φ) , one may assume $a = -1$

or $s = 1$ when $p = 1$; then

$$z \dot{u} = u + q - b_1 - b_2 z - b_4 z^3 + q z u + \frac{z}{2} u^2 .$$

To obtain the condition for stability , set $u = \alpha + z v$ and find

$$\alpha = b_1 - q , \quad \dot{\alpha} = -b_2 + \alpha q + \frac{\alpha^2}{2} ;$$

therefore ,

$$(51.2) \quad b_2 = \frac{1}{2} (b_1^2 - q^2) - \dot{b}_1 + \dot{q} .$$

For P_1 , $p = 5$, $as = 3$, one has

$$z \dot{u} = 5u + q + 3b_1 - b_2 z - \frac{b_4}{9} z^3 + qzu + \frac{zu^2}{2} .$$

The condition for stability determined by this equation is complicated ; we use another method .

A transformation (λ, φ) may always be chosen such that $b_1 + 2a_1 = 0$; then , $b_1 = -2q$ and $b_2 = 3\left(\dot{q} + \frac{q^2}{2}\right)$

[from (51.2)] . Therefore , (51.1) may be rewritten as

$$(51.3) \quad \ddot{y} = \frac{1}{2} \frac{\dot{y}^2}{y} - y\dot{y} + q\dot{y} + \frac{1}{2} y^3 - 2qy^2 + 3\left(\dot{q} + \frac{q^2}{2}\right) y - \frac{72Hr^2}{y} .$$

Now , set

$$(51.4) \quad 12 w = \dot{y} + y^2 - 3 q y$$

and note that w is regular for the parametric poles corresponding to $p = 1$ and has a double pole at the parametric poles corresponding to $p = 5$.

Equation (51.3) gives

$$(51.5) \quad y = 6 \frac{w^2 - H r^2}{w - q w}$$

so that (51.3) is equivalent to (51.4-5) ; moreover , w is stable if y is stable and conversely .

On eliminating y between (51.4-5) , one obtains

$$\ddot{w} = 6 w^2 + (\dot{q} + q^2) w - 6 H r^2$$

$$\left[\text{observe that } \dot{r} = q r , q w^2 - H r \dot{r} = q(w^2 - H r^2) \right] .$$

Because $\dot{q} + q^2 = \frac{\ddot{r}}{r}$, one finds

$$\ddot{w} = 6 w^2 + \frac{\ddot{r}}{r} w - 6 H r^2 .$$

This is an equation of type $A(x,y) = 0$ and class P2 , $p = 5$.

To obtain the canonical form of this equation , set $12 g = \frac{\ddot{r}}{r}$

and $w = v - g$ so that

$$(51.6) \quad \ddot{v} = 6 v^2 + \ddot{g} - 6 g^2 - 6 H r^2 .$$

We have to consider two cases according as to whether $H = 0$ or $H \neq 0$.

1. $H = 0$; we obtain equation E.6 . In order that (51.6) be stable , g and v must be solutions of

$$(51.7) \quad \ddot{v} = 6 v^2 + K x + K_1 .$$

If v_1 and v are respectively a particular and the general solution of (51.7) , one has $g = v_1$, $\ddot{r} = 12 v_1 r$ and

$$\frac{1}{y} = \frac{1}{6} \frac{r}{(v - v_1)^2} \frac{d}{dx} \left(\frac{v - v_1}{r} \right) ;$$

therefore , $y(x)$ is stable .

ii. $H \neq 0$; we obtain equation E.17 . We may evidently assume $H = 1$. In order that (51.6) be stable , one must have

$$(51.8) \quad \begin{cases} \ddot{g} = 6 g^2 + 6 r^2 + K x + K_1 \\ \ddot{v} = 6 v^2 + K x + K_1 \end{cases} ,$$

Because $\ddot{r} = 12 g r$, one may write

$$\ddot{g} \pm \ddot{v} = 6(g \pm r)^2 + K x + K_1 ;$$

therefore , if v , v_1 , v_2 are distinct integrals of (51.7) , one has $g + r = v_1$, $g - r = v_2$ and

$$2g = v_1 + v_2 , \quad 2r = v_1 - v_2 , \quad q = \frac{\dot{v}_1 - \dot{v}_2}{v_1 - v_2} .$$

Moreover ,

$$w = v - \frac{v_1 + v_2}{2} ,$$

$$y = 6 \frac{(v - v_1)(v - v_2)}{v - \frac{v_1 + v_2}{2} - q \left(v - \frac{v_1 + v_2}{2} \right)} ;$$

therefore , $y(x)$ is stable .

52. Equations E.7 and E.18. Consider the equation

$$(52.1) \quad \ddot{y} - \frac{1}{2} \frac{\dot{y}^2}{y} = a_1 \dot{y} + by^3 + b_1 y^2 + b_2 y + b_3 + \frac{b_4}{y} ;$$

if $b_4 \neq 0$, we have equation E.18 and if $b_3 = b_4 = 0$, equation E.7.

By a transformation (λ, φ) , we may assume $a_1 = 0$, $b = \frac{3}{2}$.

For Z1 , one has $p = 0$, $b_4 s^2 + 1 = 0$. The condition for

stability is $b_3 + \dot{s} = 0$, i.e. $b_3 = 0$, $\dot{s} = 0$ or $b_4 = K \neq 0$.

For P1, one has $p = 2$, $s = \pm 1$ and

$$zu = 2u - b_1 s - b_2 z - b_3 s z^2 - b_4 z^3 + \frac{z}{2} u^2.$$

To obtain the condition for stability, set $u = \alpha + \beta z + z^2 v$ and find

$$2\alpha - b_1 s = 0, \quad \dot{\alpha} = \beta - b_2 + \frac{\alpha^2}{2}, \quad \dot{\beta} = 0$$

so that $2\ddot{b}_1 s = -4\dot{b}_2 + b_1 \dot{b}_1$, i.e. $\ddot{b}_1 = 0$, $4\dot{b}_2 = b_1 \dot{b}_1$ or

$$b_1 = K_1 x + H, \quad 8b_2 = b_1^2 + 8K_2.$$

We have to consider two cases according as to whether $K_1 = 0$ or $K_1 \neq 0$.

a. $K_1 = 0$; then b_1, b_2, b_4 are constants and one finds the equation

$$(52.2) \quad \ddot{y} = \frac{\dot{y}^2}{2y} + \frac{3}{2} y^3 + 4K_1 y^2 + 2K_2 y - \frac{K}{2y}.$$

Using the method of variation of parameters, one obtains

$$\dot{y}^2 = y^4 + 4K_1 y^3 + 4K_2 y + K + K_3 y; \quad ;$$

$y(x)$ is an elliptic function and is stable.

[One may also multiply (52.2) by $2 \frac{y}{\dot{y}}$, integrate and take into account $2 \frac{\dot{y} \ddot{y}}{y} - \frac{\dot{y}^3}{y^2} = \frac{d}{dx} \left(\frac{\dot{y}^2}{y} \right)$.]

b. $K_1 \neq 0$. Then, by a trivial change of variable, one may assume $b_1 = 4x$ and $b_2 = 2x^2 + K_2$; the equation (50.1) becomes

$$(52.3) \quad \ddot{y} = \frac{1}{2} \frac{\dot{y}^2}{y} + \frac{3}{2} y^3 + 4xy^2 + (2x^2 + K_2)y + \frac{K}{2y}$$

which is the equation iv of Table I .

For $K = 0$, one obtains the corresponding stable equations E.7.

53. Equations E.8 and E.19. Consider the equation

$$(53.1) \quad \ddot{y} - \frac{2}{3} \frac{\dot{y}^2}{y} = \left(ay + a_1 + \frac{a_2}{y} \right) \dot{y} + \frac{3}{2} a^2 y^3 + b_1 y^2 + b_2 y + b_3 - \frac{3a_2^2}{y};$$

if $a_2 \neq 0$, we have equation E.19 and if $a_2 = b_3 = 0$, equation E.8.

By a transformation (λ, η) , we may assume $a = -\frac{2}{3}$.

Then , one has

for P1 : $p = 1$, $s = 1$ and $p = 3$, $s = -2$;

for Z1 : $p = 0$, $3a_2 s = -1$ and $p = 1$, $3a_2 s = 2$.

The condition for stability for Z1 , $p = 0$ is

$$a_1 + b_3 s - a_2 \dot{s} + \frac{2}{3} \frac{\dot{s}}{s} = 0$$

or

$$b_3 = 3 a_1 a_2 - 3 \dot{a}_2 .$$

For convenience , let us write $a_2 = r$, $3 a_1 = 2 q$ so that

$$(53.2) \quad b_3 = 2 q r - 3 \dot{r} .$$

For P1 , $p = 1$, one has

$$\dot{z}u = u + a_1 - b_1 + z(a_2 - b_2) - b_3 z^2 - b_4 z^3 + a_1 z u + a_2 z^2 u + \frac{z u^2}{3} .$$

The condition for stability is

$$\dot{a}_1 - \dot{b}_1 + a_2 - b_2 - a_1(a_1 - b_1) + \frac{1}{3}(a_1 - b_1)^2 = 0 .$$

To simplify the notations , we may assume $b_1 + 5 a_1 = 0$,
on using a transformation (λ, φ) . Then $3b_1 = -10 q$ and

$$(53.3) \quad b_2 = 4 \dot{q} + r + \frac{8}{3} q^2 .$$

Equation (53.1) becomes

$$(53.4) \quad \ddot{y} - \frac{2}{3} \frac{\dot{y}^2}{y} = - \frac{2}{3} y \dot{y} + \frac{2}{3} q \dot{y} + r \frac{\dot{y}}{y} + \frac{2}{3} y^3 - \frac{10}{3} q y^2 + \\ + b_2 y + b_3 - \frac{3 r^2}{y} ,$$

where b_2 is given by (53.3) and b_3 by (53.2) .

The conditions for stability for $Z1$, $p = 1$ and $P1$, $p = 3$ are complicated ; we use another method .

Observe that $\frac{\dot{y}}{y} + y - 4q$ is regular for the parametric poles $P1$, $p = 0$ and that $\frac{\dot{y}}{y} + \frac{3r}{y}$ is regular for the zeros $Z1$, $p = 0$. On setting

$$(53.4) \quad \frac{\dot{y}}{y} + y - 4 q + \frac{3r}{y} = v$$

one obtains

$$(53.5) \quad \dot{v} = - \frac{v^2}{3} - 2 v q + v y .$$

Equation (53.4) and the differential system (53.4-5) are thus equivalent . On eliminating y between (53.4-5) , one finds

$$\ddot{v} = \frac{2}{9} v^3 + 2 v^2 q + 4 v q^2 - 2 v \dot{q} + 3 r v ;$$

this is an equation of type $A(x,y) = 0$ which , on setting

$v = 3(w - q)$, reduces to the canonical form

(53.6) $\ddot{w} = 2 w^3 + (3r - 2\dot{q} - 2q^2)w + \ddot{q} + 2q\dot{q} - 3rq$
 [see (24.3)] . For this equation to be stable , it is necessary that

$$\begin{aligned} 3r - 2\dot{q} - 2q^2 &= Kx + H , \\ \ddot{q} + 2q\dot{q} - 3rq &= K_1 . \end{aligned}$$

On eliminating r between these relations , one obtains

$$(53.7) \quad \ddot{q} = 2q^3 + (Kx + H)q + K_1$$

equation II of Table I ; then ,

$$(53.8) \quad 3r = Kx + H + 2\dot{q} + 2q^2 .$$

The stable equations for this class are of the form

$$\begin{aligned} (53.9) \quad \ddot{y} &= \frac{2}{3} \frac{\dot{y}^2}{y} - \frac{2}{3} y\dot{y} + \frac{2}{3} q\dot{y} + r \frac{\dot{y}}{y} + \frac{2}{3} y^3 - \\ &- \frac{10}{3} qy^2 + (4\dot{q} + r + \frac{8}{3} q^2)y + (2qr - 3\dot{r}) - \frac{3r^2}{y} , \end{aligned}$$

where q is given by (53.7) and r by (53.8) .

The general integral of (53.9) is given by

$$y = \frac{\dot{w} - \dot{q} + w^2 - q^2}{w - q} ,$$

where q and w ($\neq q$) are respectively a particular solution and the general solution of (53.7) .

To obtain equation E.8 , one has to assume $r = 0$. Equation (53.6) is then

$$\ddot{w} = 2w^3 - 2(\dot{q} + q^2)w + \ddot{q} + 2q\dot{q} ;$$

for this equation to be stable , it is necessary that

$$(53.10) \quad \dot{q} + q^2 = Kx + H$$

so that $\ddot{q} + 2q\dot{q} = K$.

The stable equation E.8 is then

$$(53.11) \quad \ddot{y} = \frac{2}{3} \frac{\dot{y}^2}{y} - \frac{2}{3} y\dot{y} + \frac{2}{3} q\dot{y} + \frac{2}{3} y^3 - \frac{10}{3} qy^2 + \left(4\dot{q} + \frac{8}{3} q^2 \right) y ,$$

where q is a solution of (53.10). Its general integral is

$$y = \frac{\dot{w} - \dot{q} + w^2 - q^2}{w - q} ,$$

where w is a solution of

$$\ddot{w} = 2w^3 - 2(Kx + H)w + K .$$

54. Equations E.9 and E.20. Consider the equation

$$(54.1) \quad \ddot{y} - \frac{4}{5} \frac{\dot{y}^2}{y} = \left(ay + a_1 + \frac{a_2}{y} \right) \dot{y} + 5a^2y^3 + b_1y^2 + b_2y + b_3 - \frac{5a_2^2}{9y} ;$$

if $a_2 \neq 0$, we have equation E.20 and if $a_2 = b_3 = 0$, equation E.9.

By a transformation (λ, φ) , we may assume $a = -\frac{2}{5}$.

Then, one has

for P1 : $p = 1$, $s = 1$ and $p = 2$, $s = -\frac{3}{2}$;

for Z1 : $p = 0$, $5a_2s = -3$ and $p = 3$, $5a_2s = 12$.

We consider the simplest conditions of stability.

For Z1, $p = 0$, one finds

$$\dot{a}_2 = a_1a_2 - \frac{3}{5} b_3 .$$

For P_1 , $p = 1$, one has

$$\dot{z}u = u + a_1 - b_1 + (a_2 - b_2)z - b_3z^2 - b_4z^3 + (a_1 + a_2z)zu + \frac{z}{5}u^2.$$

On setting $u = \alpha + zv$, one obtains

$$\begin{aligned}\alpha + a_1 - b_1 &= 0, \\ \dot{\alpha} &= a_2 - b_2 + a_1\alpha + \frac{\alpha^2}{5}.\end{aligned}$$

For convenience, on using a transformation (λ, φ) , we assume $b_1 + 14a_1 = 0$ and write

$$5a_1 = -q, \quad 5b_1 = 14q, \quad a_2 = r.$$

Hence, the conditions for stability are

$$\text{for } Z_1, p = 0: \quad 3b_3 = -(qr + 5\dot{r});$$

$$\text{for } P_1, p = 1: \quad b_2 = r - 3\dot{q} + \frac{6}{5}q^2.$$

Then, equation (54.1) is

$$\begin{aligned}(54.2) \quad \ddot{y} &= \frac{4}{5} \frac{\dot{y}^2}{y} - \frac{2}{5} y\dot{y} - \frac{1}{5} q\dot{y} + r \frac{\dot{y}}{y} + \frac{4}{5} y^3 + \frac{14q}{5} y^2 \\ &+ \left[r - 3\dot{q} + \frac{6}{5} q^2 \right] y - \frac{1}{3} (qr + 5\dot{r}) - \frac{5}{9} \frac{r^2}{y}.\end{aligned}$$

The conditions for stability for P_1 , $p = 2$ and Z_1 , $p = 3$ give the values of q and r ; because these conditions are complicated, we use another method.

Set

$$\frac{\dot{y}}{y} + y + 3q + \frac{5}{3} \frac{r}{y} = -\frac{5}{2} v.$$

[Note that v is regular at the parametric poles P_1 , $p = 1$ and at the parametric zeros Z_1 , $p = 0$ and has simple poles

for $P1$, $p = 3$ and $Z1$, $p = 2$ } . The equation (54.2) is equivalent to the differential system

$$\begin{aligned}\dot{y} &= -\frac{5}{2} v y - y^2 - 3 q y - \frac{5}{3} r , \\ y &= \frac{\dot{v}}{v} - \frac{v}{2} - q .\end{aligned}$$

On eliminating y between these relations , one finds

$$\ddot{v} = - (v+q)\dot{v} + v^3 + 3 q v^2 + \left(\dot{q} + 2 q^2 - \frac{5}{3} r \right) v .$$

This equation is of type $A(x,y) = 0$ and on setting $v + q = w$ reduces to the canonical form

$$\ddot{w} = - w\dot{w} + w^3 + (2\dot{q} - q^2 - \frac{5}{3} r)w + \ddot{q} - q\dot{q} + \frac{5}{3} r q .$$

For this equation to be stable , it is necessary that

$$(54.3) \quad \begin{cases} 2 \dot{q} - q^2 - \frac{5}{3} r = -12 v_1 , \\ \ddot{q} - q\dot{q} + \frac{5}{3} r q = 12 \dot{v}_1 , \end{cases}$$

where v_1 is a solution of

$$(54.4) \quad \ddot{v} = 6 v^2 + K x + H .$$

From (54.3) , one deduces

$$(54.5) \quad \ddot{q} = - q\dot{q} + q^3 - 12 v_1 q + 12 \dot{v}_1 .$$

The general solution of (54.5) is [see (25.5)]

$$q = \frac{\dot{V} - \dot{v}_1}{V - v_1} ,$$

where $V (\neq v_1)$ is the general solution of (54.4) ; then ,

$$(54.6) \quad \frac{5}{3} r = 24 v_1 + 12 V - 3 \left(\frac{\dot{V} - \dot{v}_1}{V - v_1} \right)^2 .$$

The general solution of (62) where q and r are given by (54.5-6) , is

$$y = \frac{\dot{w} - \dot{q}}{w - q} - \frac{w}{2} - \frac{q}{2} ,$$

where $w (\neq q)$ is the general solution of (54.5) .

To obtain equation E.9 , one has to assume $r = 0$. Then , instead of (54.3) , one finds

$$2 \dot{q} - q^2 = - 12 v_1 , \quad \ddot{q} - q\dot{q} = 12 \dot{v}_1$$

so that $v_1 = K$ is a constant . The general solution of the corresponding equation E.9 is

$$y = \frac{\dot{w} - \dot{q}}{w - q} - \frac{w}{2} - \frac{q}{2} ,$$

where

$$\ddot{w} = - w \dot{w} + w^3 - 12 K w ,$$

$$2 \dot{q} = q^2 + K .$$

VIII. Equations of the type $A(x,y) = \frac{1}{2y} + \frac{1}{y-1}$.

55. The stable equations of this type are of the form

$$(55.1) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + \frac{M(x, y)}{y(y-1)} \dot{y} + \frac{N(x, y)}{y(y-1)} ,$$

where

$$\frac{M}{D} = a y + a_1 + \frac{M_0}{D} ,$$

$$\frac{N}{D} = b y^3 + b_1 y^2 + b_2 y + b_3 + \frac{N_0}{D} ;$$

$M_0(x,y)$, $N_0(x,y)$ are polynomials of the first degree in y .

The values of y for which the general existence theorem of Cauchy does not apply are $y = \infty$, 0 and 1 .

If a and b are not both zero , $y(x)$ has simple parametric poles ; if $a = b = 0$, $y(x)$ has double parametric poles .

If $y(x)$ has a simple parametric pole , set $y = \frac{s}{z}$, $\dot{z} = 1+uz$ and determine s by (note that $n = -2$)

$$2 b s^2 - 2 a s - 1 = 0 ;$$

because $p = as$ and must be an integer , one has $a = 0$, $b \neq 0$.

If $y(x)$ has a double parametric pole , one has $a = b = 0$; because $p = -1$, one also has $b_1 = 0$.

To investigate the coefficients of M and N in a neighborhood of a parametric zero , replace y by w^{-1} in (55.1) so as to obtain

$$\ddot{w} = \left(\frac{1}{2w} + \frac{1}{w-1} \right) \dot{w}^2 - w^3 M \left(x, \frac{1}{w} \right) \frac{\dot{w}}{w(w-1)} + N \left(x, \frac{1}{w} \right) \frac{w^5}{w(w-1)}.$$

This equation has the same form as (55.1). Therefore, if $y(x)$ has simple parametric zeros, then $M_0(x,0) = 0$, $N_0(x,0) \neq 0$; if $y(x)$ has double parametric zeros, then $M_0(x,0) = 0$, $N_0(x,0) = 0$ and the coefficient of y in $N(x,y)$ is also zero.

To sum up, and changing the notations, equation (55.1) may be rewritten as

$$(55.2) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + \frac{ay-b}{y-1} \dot{y} + y(y-1) Q(x,y),$$

where

$$(55.3) \quad Q(x,y) = e y + f + \frac{g}{y^2} + \frac{h}{y} + \frac{k}{(y-1)^2} + \frac{\ell}{y-1}.$$

Note that $e = 0$ implies $f = 0$ and that $g = 0$ implies $h = 0$.

The transformation $y = w - 1$ brings equation (55.2) to an equation of the same form according to the equivalence table

(y) :	a	b	e	f	g	h	k	ℓ
(w) :	b	a	g	h	e	f	k	$k-\ell$

The substitution $t = \varphi(x)$ transforms (55.2) into

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1} \right) y'^2 + \frac{y'}{y-1} [(a-\varphi)y-b+\varphi] \frac{1}{\varphi} + \frac{y(y-1)}{\varphi^2} Q(x,y).$$

Observe also that $y = 1$ does not play the same role as $y = \infty$ or $y = 0$; accordingly, $y = 1$ must be considered separately.

56. Parametric poles. Set $y = \frac{s}{z}$, $\dot{z} = 1 + u z$ and determine s by $2es^2 = 1$; then $p = 0$ and

$$z \dot{u} = \frac{\dot{s} - 1}{s} + a + (e - f) s + O(z) .$$

The condition for stability is

$$\dot{s} - 1 + as + (e - f) s^2 = 0 ;$$

therefore , $\dot{s} + a s = 0$, $(e - f) s^2 = 1$ and finally

$$(56.1) \quad \dot{e} = 2 e a ,$$

$$(56.2) \quad e + f = 0 .$$

Note that for P2 , $e = f = 0$ so that (56.2) still holds .

Parametric zeros . From the equivalence table , one deduces at once the conditions for stability

$$(56.3) \quad \dot{g} = 2 b g ,$$

$$(56.4) \quad g + h = 0 .$$

Note that for Z2 , $g = h = 0$ so that (56.4) still holds .

57. Parametric unities . We have to consider separately the parametric unities of the first order and the parametric unities of the second order .

For the parametric unities of the first order , set $y = 1 + sz$, $\dot{z} = 1 + u z$; one has

$$(57.1) \quad \begin{cases} \dot{y} = s(1 + zu + \frac{\dot{s}}{s} z) , \\ \ddot{y} = s(z\dot{u} + u + 2 \frac{\dot{s}}{s} + \frac{\ddot{s}}{s} z + 2 \frac{\dot{s}}{s} zu + zu^2) , \end{cases}$$

$$\frac{1}{2y} + \frac{1}{y - 1} = \frac{1}{sz} + \frac{1}{2} - \frac{sz}{2} + \frac{s^2 z^2}{2} + O(z^3) ,$$

$$\frac{ay - b}{y - 1} = a + \frac{a - b}{sz} ,$$

$$\begin{aligned}
 y(y-1) Q(x,y) &= \frac{k}{sz} + k + 1 + (e+f+g+h+l)sz \\
 (57.2) \quad &+ (2e + f - g)s^2z^2 + O(z^3) .
 \end{aligned}$$

Because of (56.2) and (56.4) , one finds

$$\begin{aligned}
 zu &= \frac{1}{z} \left(1 + \frac{a-b}{s} + \frac{k}{s^2} \right) + \left(1 + \frac{a-b}{s} \right) u \\
 (57.3) \quad &+ A_0 + A_1 z + u z(a+s) + O(z^2) ,
 \end{aligned}$$

[the coefficient of $z u^2$ is zero] where

$$\begin{aligned}
 A_0 &= \frac{s}{2} + a + (a-b) \frac{\dot{s}}{s^2} + \frac{k+l}{s} , \\
 A_1 &= - \frac{\ddot{s}}{s} + a \frac{\dot{s}}{s} + l - \frac{s^2}{2} + \dot{s} + \frac{\dot{s}^2}{s^2} .
 \end{aligned}$$

We determine s by setting

$$1 + \frac{a-b}{s} + \frac{k}{s^2} = 0 ;$$

$p = 1 + \frac{a-b}{s}$ must be an integer . Therefore , in order that

(55.1) be stable , one must have one or the other of the following distinct possibilities

- i. $a-b \neq 0$, $k=0$, $p=0$, $s=b-a$;
- ii. $a-b=0$, $k \neq 0$, $p=1$, $s^2+k=0$;
- iii. $a-b=0$, $k=0$.

[see (16.15) and (16.16) ; observe that $n = \infty$ and change s into $\frac{1}{s}$] .

Case iii corresponds to parametric unities of the second order .

When $a - b = 0$, one may assume $a = b = 0$ [choose φ such that $\ddot{\varphi} = a \dot{\varphi}$].

To determine the condition for stability corresponding to double parametric unities, observe that because $a = b = 0$, e and g are constants [see (56.1) and (56.3)]. The equation (55.1) is then

$$(57.4) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + y(y-1) \left[e(y-1) + g \left(\frac{1}{y^2} - 1 \right) + \frac{\ell}{y-1} \right].$$

Now, set $y = 1 + s z^2$, $\dot{z} = 1 + z u$ and determine s by $1+2s=0$; then

$$\ddot{y} z \dot{u} = u + \left(\frac{\dot{s}^2}{s^2} - \frac{\ddot{s}}{s} \right) \frac{z}{2} + o(z^3) + o(zu).$$

The condition for stability is

$$\frac{\ddot{s}}{s} - \frac{\dot{s}^2}{s^2} = 0 \quad \text{or} \quad s = H e^{Kx}$$

and finally,

$$(57.5) \quad \ell = H e^{Kx}.$$

58. Because $y = \infty$ and $y = 0$ play the same role, we have to consider the following nine distinct possibilities:

P2, Z2 and U2 or U1, $p = 0$ or U1, $p = 1$;

P1, Z2 and U2 or U1, $p = 0$ or U1, $p = 1$;

P1, Z1 and U2 or U1, $p = 0$ or U1, $p = 1$.

We consider these cases separately and begin with the last three .

58. P1;Z1;U1,p = 0. Equation (55.2) is

$$(58.1) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + \frac{ay - b}{y-1} \dot{y} + \\ + y(y-1) \left[e(y-1) + \frac{g}{y^2} (1-y) + \frac{\ell}{y-1} \right] .$$

The condition for stability is $A_0 = 0$ or because $s = b - a$,

$$(58.2) \quad b + \dot{a} - \dot{b} + \frac{1}{2} (b^2 - a^2) = 0 .$$

To simplify the notations, set

$$a = 2c, \quad b = -2d, \quad 2e = q^2, \quad 2g = r^2 ;$$

then, from (58.2), (56.1) and (56.3),

$$(58.3) \quad \begin{cases} \ell = 2[c^2 - d^2 - \dot{c} - \dot{d}] , \\ \dot{q} = 2c q, \quad \dot{r} + 2d r = 0 . \end{cases}$$

Equation (58.1) may be rewritten as

$$(58.4) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + \frac{2(cy + d)}{y-1} \dot{y} + \frac{(y-1)^2}{2} \left(q^2 y - \frac{r^2}{y} \right) \\ + 2(c^2 - d^2 - \dot{c} - \dot{d}) y .$$

To integrate equation (58.4), observe that

$$v = \frac{\dot{y}}{y-1} + \frac{2(c+d)}{y-1}$$

is regular for U1, that $v + q y$ is regular for P1 and that

$v + qy + t + 2(q+r)$ is regular for Z1. Accordingly, set

$$(58.5) \quad \frac{\dot{y}}{y-1} + \frac{2(c+d)}{y-1} + q y = -2w ;$$

on substituting in (58.4), one finds

$$(58.6) \quad \frac{y - 1}{4y} = - \frac{\dot{w} + 2 w d}{4w^2 - r^2} .$$

$$\left[\text{write } \dot{y} = - 2w(y - 1) - 2(c+d)y - qy(y - 1) \right] .$$

The differential system (58.5-6) is thus equivalent to the equation (58.4) ; if y is stable , w is also stable and conversely.

On eliminating y between (58.5) and (58.6) , we obtain an equation for w . For convenience , set

$$A = \dot{w} + 2 d w , \quad B = 4 w^2 - r^2$$

so that

$$y = \frac{B}{B + 4 A} , \quad y - 1 = - \frac{4A}{B + 4 A} .$$

Now , rewrite (58.5) as

$$\frac{\dot{y}}{y} + 2w + 2(c+d) + q(y-1) - \frac{2w}{y} = 0$$

and observe that

$$\dot{B} - 8 A w = - 4 B d ;$$

then

$$(58.7) \quad \dot{B} + 4 \dot{A} + 4 A q + 2(d - c) (B + 4A) = 0$$

or

$$\ddot{w} = - 2 w \dot{w} + a_1 \dot{w} + a_2 w^2 + a_3 w + a_4$$

i.e. an equation of type $A(x,y) = 0$ and class P_1 , $p = 1$; thus , w and y are stable .

To integrate (58.7) , observe that because $\dot{q} = 2 c q$, one also has

$$A q = q \dot{w} + 2 d q w = \frac{d}{dx}(qw) + 2(d - c)qw ;$$

hence , on setting

$$\begin{aligned} 4 Z &= B + 4 A + 4 q w \\ &= 4 \left[w + w^2 + w(q + 2d) - \frac{r^2}{4} \right] , \end{aligned}$$

(58.7) becomes

$$\dot{Z} = 2(c - d) Z .$$

Therefore , Z is determined by $Z = K q r$ and w by a Riccati equation .

59. P1;Z1;U1,p = 1. Equation (55.2) is

$$\begin{aligned} (59.1) \quad \ddot{y} &= \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + y(y-1) \left[e(y-1) + \frac{g}{y^2}(1-y) \right. \\ &\quad \left. + \frac{k}{(y-1)^2} + \frac{\ell}{y-1} \right] , \end{aligned}$$

where e and g are constants .

One has $k \neq 0$, $p = 1$, $s^2 + k = 0$ and

$$z \dot{u} = u + A_0 + A_1 z + s u z + O(z^2)$$

[the coefficient of zu^2 is zero] ; the condition for

stability is

$$\frac{d}{dx} \left[\frac{k + 2\ell}{2s} - \frac{\dot{s}}{s} + s \right] = 0$$

or

$$\frac{d}{ds} \frac{\dot{s}}{s} = 0 , \quad \frac{d}{dx} \left(\frac{k + 2\ell}{2s} + s \right) = 0 .$$

Therefore ,

$$(59.2) \quad s = H e^{Kx}$$

and $k + 2\ell + 2s^2 = K_1 s$ or

$$(59.3) \quad 2\ell = K_1 s - s^2 = K_1 s + k .$$

Now , we consider two cases according as K is or is not zero .

i. $K = 0$; then s , k , ℓ (and e , g) are constants .

Equation (55.2) is

$$(59.4) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + y(y-1) \left[H(y-1) + H_1 \frac{y-1}{y^2} + \frac{H_2}{y-1} + \frac{2H_3}{(y-1)^2} \right] .$$

This equation may be integrated by using the method of variation of parameters ; one finds

$$\dot{y}^2 = 2y(y-1)^2 \left[Hy - \frac{H_1}{y} - \frac{H_2}{y-1} - \frac{H_3}{(y-1)^2} + H_4 \right] .$$

As a particular case , one has

$$\ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2$$

whose integral is $y = Th^{-2}(Hx + H_1)$.

ii. $K \neq 0$. By a transformation $t = \varphi(x)$, one may assume $e^{Kx} = t$ and $k = 2Ht^2$, $\ell = Ht^2 + H_1 t$; hence , on writing again x instead of t , one has

$$(59.5) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + (y-1)^2 \left[H_2 y + \frac{H_3}{y} \right] + H_1 xy + Hx^2 \frac{y(y+1)}{y-1} .$$

This is the irreducible equation IV of Table I.

A transformation $t = \varphi(x)$ brings this equation to

$$(59.6) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 - \frac{\dot{y}}{x} + \frac{(y-1)^2}{x^2} \left(H_2 y + \frac{H_3}{y} \right) + \frac{H_1 y}{x} + \frac{Hy(y+1)}{y-1} .$$

60. P1;Z1;U2. Equation (55.2) is

$$(60.1) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + y(y-1) \left[e(y-1) + \frac{g}{y^2}(1-y) + \frac{\ell}{y-1} \right],$$

where e and g are constants .

The condition for stability is $\ell = H e^{Kx}$ [see (57.5)] .

Equation (60.1) is thus a particular case of (59.4) or (59.5).

61. On using the preceding results , the other cases are readily disposed of .

P2;Z2;U2. One has , for P2 , $e = f = 0$; for Z2 , $g = h = 0$; for U2 , $a = b = k = 0$. Equation (55.2) is

$$(61.1) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + \ell y$$

with the condition for stability $\ell = H e^{Kx}$; (61.1) is a particular case of (55.4-5) .

P2;Z2;U1,p = 0 . Equation (55.2) is

$$(61.2) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + \frac{ay-b}{y-1} \dot{y} + \ell y$$

and is a particular case of (58.1) .

P2;Z2;U1,p = 1 . Equation (55.2) is

$$(61.3) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + y \left(\frac{k}{y-1} + \ell \right)$$

and is a particular case of (59.1) .

P1;Z2;U2. Equation (55.2) is

$$(61.4) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + y(y-1) \left[e(y-1) + \frac{\ell}{y-1} \right]$$

where e is a constant . This equation is a particular case of (60.1) .

P1;Z2;U1,p = 0. Equation (55.2) is

$$(61.5) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + \frac{ay-b}{y-1} \dot{y} + y(y-1) \left[e(y-1) + \frac{\ell}{y-1} \right],$$

where $\dot{e} = 2 e a$. This equation is a particular case of (58.1).

P1;Z2;U1,p = 1. Equation (55.2) is

$$(61.6) \quad \ddot{y} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \dot{y}^2 + y(y-1) \left[e(y-1) + \frac{k}{(y-1)^2} + \frac{\ell}{y-1} \right]$$

which is a particular case of (59.1) .

IX. Equations of the type $A(x,y) = \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right) .$

62. The stable equations of this type are

$$(62.1) \quad \ddot{y} = \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + P(x,y) \dot{y} + 3y(y-1) Q(x,y) ,$$

where

$$(62.2) \quad P(x,y) = ay + \frac{b}{y} - \frac{c}{y-1} + d ,$$

$$(62.3) \quad Q(x,y) = ey + \frac{f}{y^2} - \frac{g}{(y-1)^2} + h + \frac{k}{y} + \frac{\ell}{y-1} .$$

The values of y for which the general existence theorem does not apply are $0, 1, \infty$.

The transformation $y = u^{-1}$ [resp. $y = 1 - w$] changes $y = 0, \infty, 1$ into $u = \infty, 0, 1$ [resp. $w = 1, \infty, 0$] and brings equation (62.1) to an equation of the same form according to the equivalence table

(y) :	a	b	c	d	e	f	g	h	k	ℓ
(u) :	b	a	-c	d+c	f	e	g	k	h	$-\ell-g$
(w) :	-a	c	b	a+d	e	g	f	-e-h	ℓ	k

Therefore, the values $y = 0, 1, \infty$ play the same role.

Note also that the transformation defined by $(1-y)u = 1$ brings equation (62.1) to an equation of the same form according to the table of equivalence

$$\begin{array}{lcl}
 (y) : & a & b \quad c \quad d \quad e \quad f \quad g \quad h \quad k \quad \ell \\
 (u) : & c & -a \quad -b \quad a+b+d \quad g \quad e \quad f \quad \ell \quad -h-e \quad -f-h .
 \end{array}$$

The solution $y(x)$ of (62.1) has simple parametric poles if a and e are not both zero ; $y(x)$ has double parametric poles if $a = e = 0$.

Suppose $y(x)$ has a simple parametric pole . Set $y = \frac{s}{z}$, where s is given by $3es^2 - as - \frac{2}{3} = 0$; then $n = -3$ and $p = \frac{1}{3} + as$.

In order that $y(x)$ be stable , one must have $e = 3a^2$ and $p = 0$, $3as + 1 = 0$ or $p = 1$, $3as = 2$.

If $y(x)$ has a double parametric pole and is stable , then $p = -\frac{1}{3}$ so that $h = 0$. [Note that when $a = e = h = 0$,

$y(x)$ has in fact parametric poles of the third order] .

To sum up , according to the equivalence tables , one has

for P1 : $e = a^2$;	for P2 : $a = e = h = 0$;
for Z1 : $f = b^2$;	for Z2 : $b = f = k = 0$;
for U1 : $g = c^2$;	for U2 : $c = g = \ell = 0$	

Therefore ,

$$(62.4) \quad Q(x,y) = a^2y + \frac{b^2}{y^2} - \frac{c^2}{(y-1)^2} + h + \frac{k}{y} + \frac{\ell}{y-1} ,$$

where $h = 0$ if $a = 0$; $k = 0$ if $b = 0$; $\ell = 0$ if $c = 0$.

The transformation $t = \varphi(x)$ brings equation (62.1) to

$$(62.5) \quad y'' = \frac{3}{2} \left(\frac{1}{y} + \frac{1}{y-1} \right) y'^2 + \left(P(x,y) - \ddot{\varphi} \right) \frac{y'}{\dot{\varphi}} + \frac{3y(y-1)}{\dot{\varphi}^2} Q(x,y) ;$$

on choosing φ such that $\ddot{\varphi} = d$, one may assume $d = 0$.

63. Because $y = \infty$, 0 , 1 play the same role, we have to consider only four distinct possibilities, namely
 $(P2;Z2;U2)$, $(P2;Z2;U1)$, $(P2;Z1;U1)$, $(P1;Z1;U1)$.
 Equation (62.2) corresponding to $(P2;Z2;U2)$ is

$$\ddot{y} = \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2$$

for one may assume $d = 0$ by a proper choice of φ . Then, y is given by

$$\dot{y}^3 = K y^2 (y-1)^2;$$

$y(x)$ is thus an elliptic function and is stable.

[On setting $2y = 1 + u$, one has $\dot{u}^3 = K(u^2 - 1)^2$].

For the other cases, at least one of a , b , c is not zero. We may assume $c \neq 0$. Indeed, the transformations $y = u$, $y = u^{-1}$, $y = 1 - u$ and their products $y = \frac{u-1}{u}$, $y = \frac{1}{1-u}$, $y = \frac{u}{u-1}$ bring equation (62.1) to equations of the same form, and according to the equivalence tables, commute a , b , c . We consider first the equation $(P1;Z1;U1)$.

64. For a simple pole of $y(x)$, set $y = \frac{s}{z}$, $\dot{z} = 1 + uz$ and note that

$$P(x,y) = a \frac{s}{z} + d + (b-c) \frac{z}{s} + O(z^2),$$

$$Q(x,y) = a^2 \frac{s}{z} + h + (k + \ell) \frac{z}{s} + O(z^2);$$

then, substitute in (62.1), determine s by $3a^2 s^2 - as - \frac{2}{3} = 0$

and set $p = \frac{1}{3} + as$ so that

$$z \dot{u} = pu + A + Bz + Cz - \frac{1}{3} zu^2 + O(z^2) ,$$

where

$$\begin{aligned} A &= \frac{2}{3} \frac{\dot{s}}{s} - \frac{2}{3s} + d - a \dot{s} + 3(a^2 - h) s , \\ B &= \frac{d}{dx} \left(\frac{\dot{s}}{s} \right) - \frac{1}{3} \frac{\dot{s}^2}{s^2} - \frac{2}{3s^2} + \frac{4}{3} \frac{\dot{s}}{s^2} + \frac{b-c}{s} - d \frac{\dot{s}}{s} + 3(h-k-\ell) , \\ C &= \frac{2}{3} \frac{\dot{s}}{s} - \frac{4}{3s} + d . \end{aligned}$$

i. Suppose $p = 0$, $3as + 1 = 0$. The condition for stability is $A = 0$ or

$$(64.1) \quad h + a(a + d) - \dot{a} = 0 .$$

From the equivalence tables , one has the two other conditions for stability corresponding to Z1 , $p = 0$ and U1 , $p = 0$ respectively

$$(64.2) \quad \begin{cases} k + b(b + c + d) - \dot{b} = 0 , \\ \ell + c(a + b + c + d) - \dot{c} = 0 . \end{cases}$$

ii. On using the relations (64.1-2) , we shall reduce our problem to another of the same type involving an equation with only two singular values 0 and ∞ . To do this , observe that the parametric zeros and the parametric poles of $y(x)$ are poles of the first order for $\frac{\dot{y}}{y}$; hence , the singular values of $\frac{\dot{y}}{y}$ are ∞ and the value corresponding to $y = 1$.

Now , $\frac{\dot{y}}{y} - 3ay$ is regular for the parametric poles $P1$,
 $p = 0$; $\frac{\dot{y}}{y} + 3 \frac{b}{y}$ is regular for the parametric zeros $Z1$,
 $p = 0$ and $\frac{\dot{y}}{y} - 3c$ is regular for the parametric unities $U1$,
 $p = 0$. Therefore , $\frac{\dot{y}}{y} - 3ay + \frac{3b}{y} - 3c$ is regular for $P1$,
 $p = 0$; $Z1$, $p = 0$; $U1$, $p = 0$.

Accordingly , set

$$(64.3) \quad \frac{\dot{y}}{y} = 3a(y - 1) + 3b \left(1 - \frac{1}{y} \right) + 3c + 3v ;$$

equation (62.1) gives

$$(64.4) \quad y - 1 = \frac{Q(v)}{v - P(v)} ,$$

where

$$(64.5) \quad \begin{cases} P(v) = v^2 + (2a + 2b + 2c + d) v , \\ Q(v) = 2v^2 + 3cv . \end{cases}$$

Equation (62.1) and the differential system (64.3-4) are thus equivalent ; if y is stable , v is also stable and conversely .

To determine v , one has to eliminate y between (64.3) and (64.4) . To do this , rewrite (64.3) as

$$\frac{\dot{y}}{y - 1} = 3a(y - 1) + \frac{3(v + c)}{y - 1} + 3(a + b + c + v) ,$$

take the logarithmic derivative of (64.4) and observe that

$$\frac{3(v + c)}{Q(v)} = \frac{1}{v} + \frac{1}{2v + 3c} ;$$

then

$$\frac{\ddot{v} - \dot{P} + 3aQ}{\dot{v} - P} = \frac{\dot{v} + 3\dot{c}}{2v + 3c} + \frac{P}{2v + 3c} - 2v - (a+b+c-d),$$

i.e., v is a solution of an equation

$$(64.6) \quad \ddot{v} = \frac{\dot{v}^2}{2v + 3c} + B(x;v) \dot{v} + C(x;v)$$

of type $A(x;v) = \frac{1}{2v}$.

To reduce (64.6) to a canonical form, set

$$u = v + \frac{3c}{2},$$

$$(64.7) \quad \alpha = 2(a+b) - c + d,$$

$$(64.8) \quad \beta = 3c(a+b) + \frac{3c}{4}(c+2d) - \frac{3\dot{c}}{2}$$

so that

$$\dot{v} - P = \dot{u} - u^2 - \alpha u + \beta,$$

$$Q(v) = 2u \left(u - \frac{3c}{2} \right).$$

A simple calculation shows that

$$(64.9) \quad \ddot{u} = \frac{\dot{u}^2}{2u} + \frac{3u^3}{2} + (a+b+c+2d) \dot{u} + 3(b-a-c) u^2 + u \left[\alpha + 9ac - \beta - \frac{3\alpha}{2}(c+d) \right] + \beta(a+b+c+2d) - \beta - \frac{\beta^2}{2u}.$$

Now, the transformation $t = \varphi(x)$ brings equation (62.1) to (62.5). If u is replaced by $w \dot{\varphi}(x)$, (64.3) and (64.4) remain invariant if the coefficients of (62.1) are replaced by the corresponding coefficients of (62.5). If $\varphi(x)$ is chosen

so that $2 \ddot{\varphi} = (a+b+c+2d) \dot{\varphi}$, and if w is to be stable (and thus also y), then equation (64.9) must reduce to the canonical form [see § 52 , eq. E.7 and E.18]

$$(64.10) \quad \ddot{w} = \frac{\dot{w}^2}{2w} + \frac{3w^3}{2} + 4 D w^2 + 2 E w - \frac{F^2}{2w},$$

where D , E , F are given by (52.2) and (52.3) .

Therefore , to determine a , b , c , d , one has

$$(64.11) \quad \alpha = 2(a + b) - c + d ,$$

$$(64.12) \quad \beta = 3c(a+b) + \frac{3c}{4} (c+2d) - \frac{3c}{2} ,$$

$$(64.13) \quad a + b + c + 2d = 0 ,$$

$$(64.14) \quad 3(b - c - a) = 4 D ,$$

$$(64.15) \quad \dot{\alpha} + 9 a c - \beta - \frac{3\alpha}{2} (c + d) = 2 E ,$$

$$(64.16) \quad \dot{\beta} = 0 ,$$

$$(64.17) \quad \beta = F .$$

Set $3c = 2 W$. Then , from (64.12-13) and (64.17) ,

$$(64.18) \quad 3(a + b) = 2 \frac{\dot{W}}{W} + \frac{2F}{W} ;$$

from (64.14) and (64.13) respectively ,

$$(64.19) \quad \begin{aligned} 3(b - a) &= 4D + 2 W , \\ 3 d &= - \left(\frac{\dot{W}}{W} + \frac{F}{W} + W \right) . \end{aligned}$$

Therefore ,

$$\alpha = 3(a + b + d) = \frac{\dot{W}}{W} + \frac{F}{W} - W$$

so that , from (64.15) ,

$$(64.20) \quad \ddot{W} = \frac{\dot{W}^2}{2W} + \frac{3}{2} W^3 + 4 D W^2 + 2 E W - \frac{F^2}{2W} .$$

65. From the preceding , it follows that the stable equations of the class (P1,Z1,U1) are

$$\ddot{y} = \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + \left(ay + \frac{b}{y} - \frac{c}{y-1} + d \right) \dot{y} \\ + 3y(y-1) \left[a^2 y + \frac{b^2}{y} - \frac{c^2}{(y-1)^2} + h + \frac{k}{y} + \frac{\ell}{y-1} \right] ,$$

where

$$3a = \frac{\dot{W}}{W} + \frac{F}{W} - 2D - W , \quad 3b = \frac{\dot{W}}{W} + \frac{F}{W} + 2D + W ,$$

$$3c = 2W , \quad 3d = - \left(\frac{\dot{W}}{W} + \frac{F}{W} + W \right) ,$$

$$h = \dot{a} + \frac{a}{2} (b + c - a) ,$$

$$k = \dot{b} - \frac{b}{2} (b + c - a) ,$$

$$\ell = \dot{c} - \frac{c}{2} (a + b + c) ;$$

W is a solution of (64.20) and

$$y - 1 = \frac{2w \left(w - \frac{3c}{2} \right)}{w - w^2 - \alpha w + \beta} = \frac{2w(w - W)}{w - W - (w-W)(w+W + \alpha)}$$

because

$$\beta = F = (\alpha + W)W - \dot{W} .$$

Therefore ,

$$(65.1) \quad y = \frac{\dot{w} - \dot{W} + (w - W)(w - W - \alpha)}{w - W - (w-W)(w+W + \alpha)}$$

and is stable .

66. Remark . For P_1 , $p = 1$, $3as = 2$, one has

$$z \dot{u} = u + A + Bz + Czu - \frac{1}{3} zu^2 + O(z^2) ,$$

where

$$D = \frac{h}{a} + a + d \quad \text{so that} \quad \dot{a} = a D , \quad (\text{see } 64.1) ;$$

$$A = D - \frac{3h}{a} , \quad 3C = 3d - 6d - 2D ,$$

$$B = -\dot{D} - \frac{1}{3} D^2 - (2a - d)D + \frac{3}{2} a(b-c-a) + 3(h - k - \ell) .$$

The condition for stability is

$$\dot{A} + B - AC - \frac{A^2}{3} = 0$$

or

$$\frac{d}{dx} \frac{h}{a} - \frac{1}{2} a(b-c-a) + k + \ell - \frac{2}{a} hd + \frac{h}{a} D = 0$$

or

$$2(a\dot{h} - 2a\dot{h}) + (a^2 + 2h)^2 - a^3(b-c) + 2a^2(k + \ell) = 0 .$$

There are two similar conditions for stability for Z_1 , U_1 ,
namely

$$2(b\dot{k} - 2k\dot{b}) + (b^2 + 2k)^2 - b^2(a+c) + 2b^2(h - \ell - c^2) = 0 ,$$

$$2(c\dot{\ell} - 2\ell\dot{c}) + (c^2 + 2\ell)^2 - c^2(b-a) - 2c^2(h+k+a^2+b^2) = 0 .$$

These relations are not easy to handle and thus the reason
why another method was used .

67. P2;Z2;U1. One has $a = h = 0$ for P2 , $b = k = 0$ for Z2 ;
the stable equation for this class is

$$(67.1) \quad \ddot{y} = \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + \left(d - \frac{c}{y-1} \right) \dot{y} + 3y \left(1 - \frac{c^2}{y-1} \right),$$

where

$$\ell = \dot{c} - c(c + d).$$

We use the above method and assume $c = 2d = 0$; then w is
determined by (64.10) and c and d by (64.11-17) . One easily
finds

$$\begin{aligned} 3c &= 2W, & \dot{W} + F &= 0, & 2D + W &= 0, & 3d &= -W, \\ \alpha &= -W, & \beta &= F, & E &= D^2; \end{aligned}$$

therefore ,

$$W = -F x - H.$$

Equation (67.1) is then

$$\ddot{y} = \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 - \frac{W}{3} \cdot \frac{y+1}{y-1} - 2y \left[F + \frac{W^2}{3} \cdot \frac{y+1}{y-1} \right];$$

its general solution is

$$\begin{aligned} y &= 1 + \frac{2w(w-W)}{w - w^2 + Ww + F} = \frac{\dot{w} + w^2 - Ww + F}{w - w^2 + Ww + F} \\ &= \frac{\dot{w} - \dot{W} + w(w-W)}{w - W - w(w-W)}, \end{aligned}$$

where w is determined by

$$\ddot{w} = \frac{\dot{w}^2}{2w} + \frac{3w^3}{2} - 2Ww^2 + 2D^2w - \frac{F^2}{2w}.$$

68. P2;Z1;U1. One has $a = h = 0$ for P2 . The stable equation for this class is equation (62.1) , where $a = h = 0$. We use again the above method and assume

$$b + c + 2 d = 0 \quad , \quad 3 c = 2 W .$$

Then , one has

$$(68.1) \quad \dot{W} = W^2 + 2 D W - F$$

so that W is determined by a Riccati equation .

Moreover ,

$$\begin{aligned} 3 b &= 4 D + 2 W \quad , \quad 3 d = - 2 W - 2 D \quad , \\ \alpha &= 2 D \quad , \quad \beta = F \end{aligned}$$

and

$$2 \dot{D} + 2 D^2 - F = 2 E .$$

The stable equation is thus

$$\begin{aligned} \ddot{y} &= \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + \left(\frac{b}{y} - \frac{c}{y-1} + d \right) \dot{y} + \\ &+ 3y(y-1) \left[\frac{b^2}{y^2} - \frac{c^2}{(y-1)^2} + \frac{k}{y} + \frac{\ell}{y-1} \right] \quad , \end{aligned}$$

where $k = \dot{b} + b d$, $\ell = \dot{c} + c d$.

The general solution is given by (65.1) , where W is a solution of the Riccati equation (68.1) and where w is the general solution of (64.10) .

Note that if W is replaced by $-\frac{\dot{u}}{u}$ in (68.1) , one

obtains

$$\ddot{u} - 2x \dot{u} - Fu = 0 \quad ;$$

if $F = -2n$, n an integer, this equation is satisfied by $H_n(x)$, the Hermite polynomial of order n .

X. Equations of the type $A(x,y) = \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) .$

69. The stable equations of this type are

$$(69.1) \quad \ddot{y} = \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + P(x,y) \dot{y} + y(y-1) Q(x,y) ,$$

where $P(x,y)$, $Q(x,y)$ are given by (62.2-3).

The values of y for which the general existence theorem does not apply are 0 , 1 , ∞ .

The solution $y(x)$ of (69.1) may have simple or double parametric poles .

If $y(x)$ has a simple parametric pole , set $y = \frac{s}{z}$,
 $\dot{z} = 1 + uz$, where s is given by $2es^2 = 1$; then $n = -2$ (case vi),
 $a = 0$, $e \neq 0$ and $p = 0$. Then , substitution in (69.1) yields

$$z\dot{u} = \frac{\dot{s}}{s} - \frac{3}{4s} + d + (e - h)s + O(z) .$$

The condition for stability is thus

$$\dot{s} - \frac{3}{4} + d s + (e - h) s^2 = 0$$

or , because $2es^2 = 1$,

$$\dot{s} + d s = 0 , \quad 4(e - h) s^2 = 3 ;$$

finally

$$(69.2) \quad 2h + e = 0 , \quad \dot{e} = 2 e d .$$

If $y(x)$ has a double parametric pole , then $a = e = 0$
and , as a consequence , $h = 0$.

Changing the notations and taking into account the relations (69.2) , we shall rewrite equation (69.1) as

$$(69.3) \quad \ddot{y} = \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + P^*(x,y) \dot{y} + y(y-1) Q^*(x,y) ,$$

where

$$P^*(x,y) = a + \frac{b}{y} - \frac{c}{y-1} ,$$

$$Q^*(x,y) = 4d^2(2y-1) + \frac{f}{y^2} - \frac{g}{(y-1)^2} + \frac{k}{y} + \frac{\ell}{y-1} .$$

Note that

$$(69.4) \quad \dot{d} = a \, d \, x$$

according to (69.2) ; for P1 , one has $d \neq 0$ and for P2 , $d = 0$.

The exceptional values $y = 0$ and $y = 1$ play the same role ; indeed , on setting $y = 1 - w$ in (69.3) , one obtains an equation of the same form according to the equivalence table

(y) :	a	b	c	d	f	g	k	ℓ
(w) :	a	c	b	d	g	f	ℓ	k .

It is readily seen that $y(x)$ has simple parametric zeros if b and f are not both zero and double parametric zeros if $b = f = 0$ and $k \neq 0$.

In agreement with the equivalence table , one sees at once that $y(x)$ has simple parametric unities if c and g are not both zero and double parametric unities , if $c = g = 0$ and $\ell \neq 0$. Therefore , we have to consider only six distinct possibilities namely

$$\begin{aligned} & (P_2; Z_2; U_2) \quad , \quad (P_2; Z_2; U_1) \quad , \quad (P_2; Z_1; U_1) \quad , \\ & (P_1; Z_2; U_2) \quad , \quad (P_1; Z_2; U_1) \quad , \quad (P_1; Z_1; U_1) \quad . \end{aligned}$$

We begin by considering the case $(P_1; Z_1; U_1)$.

Furthermore , note that a transformation $t = \varphi(x)$ brings equation (69.3) to

$$y'' = \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) y'^2 + [P^*(x,y) - \ddot{\varphi}] \frac{y'}{\dot{\varphi}} + \frac{y(y-1)}{\dot{\varphi}^2} Q^*(x,y)$$

70. Suppose that $y(x)$ has simple parametric zeros . Set $y = sz$, $\dot{z} = 1 + uz$ so that

$$P^*(x,y) = \frac{b}{sz} + a + c + csz + O(z) \quad ,$$

$$\begin{aligned} Q^*(x,y) &= \frac{f}{s^2 z^2} + \frac{k}{sz} - (4d^2 + g + \ell) + sz(8d^2 - 2g - \ell) \\ &\quad - s^2 z^2(3g + \ell) + O(z^3) . \end{aligned}$$

On substituting in (69.3) , one obtains

$$(70.1) \quad z \dot{u} = \frac{1}{z} \left(\frac{3}{4} + \frac{b}{s} - \frac{f}{s^2} \right) + \left(\frac{1}{2} + \frac{b}{s} \right) u + A + O(z)$$

where

$$(70.2) \quad A = a + c + b \frac{\dot{s}}{s^2} - \frac{k}{s} + \frac{f}{s} - \frac{1}{2} \frac{\dot{s}}{s} - \frac{3s}{4} .$$

Now , determine s by

$$\frac{f}{s^2} - \frac{b}{s} - \frac{3}{4} = 0 \quad ;$$

then $p = \frac{1}{2} + \frac{b}{s}$ must be an integer . Therefore , according

to Appendix I , vi [substitute s with s^{-1}] , one finds

$$(70.3) \quad f = b^2$$

and

$$p = 0, \quad 2b + s = 0; \quad p = 2, \quad 2b = 3s.$$

For Z_1 , $p = 0$, the condition for stability is $A = 0$, $s = -2b$, i.e.,

$$(70.4) \quad k = 2\dot{b} - 2b(a + b + c).$$

In agreement with the equivalence table, one obtains for U_1 , $p = 0$, the condition for stability

$$(70.5) \quad \begin{cases} g = c^2, \\ \ell = 2\dot{c} - 2c(a + b + c). \end{cases}$$

71. Before considering the case Z_1 , $p = 2$, we find the condition of stability for Z_2 .

For a double parametric zero, one has

$$(71.1) \quad b = f = 0, \quad k \neq 0.$$

On setting $y = s z^2$, $\dot{z} = 1 + u z$, one finds

$$a - \frac{c}{y - 1} = a + c + c s z^2 + O(z^4),$$

$$Q^*(x, y) = \frac{k}{s z^2} - (\ell + g + 4d^2) + O(z^2)$$

and

$$z \dot{u} = \frac{1}{2z} \left(1 - \frac{k}{s} \right) + a + c - \frac{1}{2} \frac{\dot{s}}{s} + O(z)$$

[Note that $p = 0$].

Determine s by $s = k$; because $p = 0$, the condition for stability is

$$(71.2) \quad \dot{k} = 2k(a + c) .$$

For a double parametric unity , the condition for stability is , according to the equivalence table ,

$$(71.3) \quad c = g = 0 \quad , \quad \ell \neq 0 \quad , \quad \dot{\ell} = 2 \ell (a + b) .$$

72. P1;Z1;U1. We have $d \neq 0$ for P1 . On using the conditions (70.3-5) , we reduce the problem to another one involving an equation with only one singular value , namely ∞ .

To do this , note that $\frac{\dot{y}}{y} + \frac{2b}{y}$ is regular for Z1 , $p = 0$

and that $\frac{\dot{y} - 2c}{1 - y}$ is regular for U1 , $p = 0$ [set $1 - y = sz$

and note that $s = -2c$. We don't pay particular attention to P1 because one ought to consider $\frac{\dot{y}}{y} + \frac{y}{s}$ with $s = \pm \frac{1}{4d}$;

$\frac{\dot{y}}{y} + 4dy$ is regular for a set of parametric poles and $\frac{\dot{y}}{y} - 4dy$ is regular for the other set of parametric poles] .

Now , set

$$\frac{\dot{y} + 2b}{y} + \frac{\dot{y} - 2c}{1 - y} = 4 v \quad ;$$

on substituting in (69.3) , one finds

$$(2y - 1) (v^2 - d^2) = \dot{v} - a v .$$

Hence , the differential system

$$(72.1) \quad \dot{y} = 2b(y - 1) + 2 c y - 4 v y(y - 1) \quad ,$$

$$(72.2) \quad 2y - 1 = \frac{\dot{v} - a v}{v^2 - d^2}$$

is equivalent to equation (69.3) ; if $y(x)$ is stable , v is also stable and conversely .

To determine v , eliminate y between (71.1-2) . To do this , set

$$\dot{v} - a v = A \quad , \quad v^2 - d^2 = B$$

and note that $\dot{d} = a d$ [see (69.4)] and

$$\dot{B} - 2 v A = 2 a B \quad ;$$

then ,

$$\dot{A} = 2(a+b+c)A + 2(v - b + c)B$$

or

$$(72.3) \quad \ddot{v} = (3a + 2b + 2c) \dot{v} + 2v^3 + 2(c - b)v^2 + \dot{a} - 2a(a + b + c) - 2d^2v + 2d^2(b - c)$$

i.e. , an equation of type I [see equation 20.5 and § 24] .

$$\text{Now , by the method used in type } \frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right)$$

[i.e. , a proper choice of φ] , one may assume

$$(72.4) \quad 3a + 2b + 2c = 0$$

so that , on setting $c - b = 3D$, one may rewrite (71.3) as

$$\ddot{v} = 2v^3 + 6Dv^2 + (\dot{a} + a^2 - 2d^2)v - 6Dd^2 .$$

The transformation $v = w - D$ brings this equation to the canonical form

$$(72.5) \quad \ddot{w} = 2w^3 + (\dot{a} + a^2 - 2d^2 - 6D^2)w + \ddot{D} + 4D^3 - (\dot{a} + a^2 - 2d^2)D - 6Dd^2 .$$

In order that w be stable , equation (72.5) must be identical to

$$(72.6) \quad \ddot{V} = 2V^3 + S V + T$$

so that

$$(72.7) \quad \dot{a} + a^2 - 2d^2 - 6D^2 = S, \quad ,$$

$$(72.8) \quad \ddot{D} + 4D^3 - (\dot{a} + a^2 - 2d^2) D - 6 D d^2 = T.$$

On using $\dot{d} = a d$, one obtains from (72.7) and (72.8) after simple calculations

$$(72.9) \quad \ddot{d} - 2d^3 - S d - 6 D^2 d = 0, \quad ,$$

$$(72.10) \quad \ddot{D} - 2 D^3 - D S - T - 6 D d^2 = 0.$$

From (72.9-10), it is clear that $D - d = V_1$ and $D + d = V_2$ are solutions of (72.6); therefore, on using (69.4) and (72.4), one finds

$$(72.11) \quad \left\{ \begin{array}{l} 2 d = V_2 - V_1 \neq 0, \quad a = \frac{\dot{V}_2 - \dot{V}_1}{V_2 - V_1}, \\ D = c - b = \frac{1}{2}(V_1 + V_2), \quad b + c = -\frac{3}{2} \frac{\dot{V}_2 - \dot{V}_1}{V_2 - V_1}. \end{array} \right.$$

The general solution of equation (69.3), where

$$(72.12) \quad \begin{array}{l} f = b^2, \quad g = e^2, \\ k = 2 \dot{b} + a b, \quad \ell = 2 \dot{c} + a c, \end{array}$$

is thus given by

$$(72.13) \quad \begin{aligned} 2 y - 1 &= \frac{\dot{w} - \dot{D} - a(w - D)}{(w - D - d)(w - D + d)} \\ &= \frac{2 \dot{V} - \dot{V}_1 - \dot{V}_2 - a(2V - V_1 - V_2)}{2(V - V_1)(V - V_2)}, \end{aligned}$$

where V is the general solution of (72.6); therefore y is stable.

73. P2;Z1;U1. For P2 , one has $d = 0$; then (72.2) becomes

$$(73.1) \quad 2y - 1 = \frac{\dot{v} - a v}{v^2} .$$

On eliminating y between (72.1) and (73.1) , one finds again (72.3) , where $d = 0$.

The method used above shows that w defined by $v = w - D$, $3D = c - b$, $3a + 2b + 2c = 0$ satisfies

$$\ddot{w} = 2 w^3 + (\dot{a} + a^2 - 6D^2)w + \ddot{D} + 4D^3 - (\dot{a} + a^2)D .$$

In order that w be stable , this equation must be identical to (72.6) so that

$$(73.2) \quad \dot{a} + a^2 - 6 D^2 = S ,$$

$$(73.3) \quad \ddot{D} + 4 D^3 - (\dot{a} + a^2) D = T .$$

Hence , D is a solution V_1 of (72.6) and a is given by the Riccati equation

$$\dot{a} + a^2 = S + 6 V_1^2$$

or by the linear equation (set $a = \frac{\dot{u}}{u}$) ,

$$(73.4) \quad \ddot{u} = (S + 6 V_1^2) u .$$

The general solution of the corresponding equation (69.1) is given by (72.13) and is stable .

74. P2 ; Z2 ; U2 . Because $d = k = \ell = 0$, the stable equation for this class is

$$(74.1) \quad \ddot{y} = \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + a \dot{y} .$$

By a transformation φ , one may assume $a = 0$. The solution of (74.1) is given by

$$\dot{y}^4 = K y^3 (y - 1)^3 ;$$

therefore, $y(x)$ is an elliptic function and is stable.

75. P1 ; Z2 ; U2 . The stable equation for this class is

$$(75.1) \quad \ddot{y} = \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + ay + y(y-1) \left[4d^2(2y-1) + \frac{k}{y} + \frac{\ell}{y-1} \right] .$$

By a transformation φ , one may assume $a = 0$; consequently, d, k, ℓ are constants.

To find the solution of (75.1), set $y = w^{-1}$ and obtain

$$(75.2) \quad \ddot{w} = \left[\frac{1}{2w} + \frac{3}{4(w-1)} \right] \dot{w}^2 + (w-1) \left[4d^2 \left(\frac{2}{w} - 1 \right) + kw - \frac{\ell w}{w-1} \right] .$$

The solution of

$$\ddot{w} = \left[\frac{1}{2w} + \frac{3}{4(w-1)} \right] \dot{w}^2$$

is

$$(75.3) \quad \dot{w}^2 = K w(w-1)^{3/2} .$$

On using the method of variation of parameters, one determines K as a function of w by

$$\frac{dK}{dw} = \frac{2}{(w-1)^{1/2}} \left[\frac{4d^2}{w^2} (2-w) + k - \frac{\ell}{w-1} \right] .$$

Now, set $w = 1 + u^2$ and $\mathcal{K}_0(u) = K(1 + u^2)$; then,

$$\frac{d\mathcal{K}_0}{du} = 4 \left[4d^2 \cdot \frac{1+u^2}{(1-u^2)^2} + k + \frac{\ell}{u^2} \right]$$

and

$$\mathcal{K}_0(u) = 4 \left[k u + \frac{\ell}{u} + \frac{4 d^2 u}{1 + u^2} + H \right] .$$

Thus , u is an elliptic function and $y(x)$ is stable .

76. P1 ; Z2 ; U1 . For convenience , we slightly change the notations and replace a by $2a$ in $P^*(x,y)$. Then , the conditions for stability are

- i. for P1 , $\dot{d} = 2 a d$,
- ii. for Z2 , $b = f = 0$, $k \neq 0$,
 $\dot{k} = 2k(2a + c)$,
- iii. for U1 , $\dot{\ell} = 2\dot{c} - 2c(2a + c)$

so that it remains only to consider the condition for stability for U1 , $p = 2$.

Equation (69.3) may be rewritten as

$$\begin{aligned} \ddot{y} = & \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + \left(2a - \frac{c}{y-1} \right) \dot{y} \\ (76.1) \quad & + y(y-1) \left[4d^2(2y-1) - \frac{c^2}{(y-1)^2} + \frac{k}{y} + \frac{\ell}{y-1} \right] . \end{aligned}$$

Now , set

$$(76.2) \quad \dot{y} = 2 c y - 4 d y (y-1) - 2v(y-1)$$

and substitute in (76.1) ; on taking into account the values of \dot{d} and $\dot{\ell}$, one obtains

$$(76.3) \quad \frac{3v^2}{y} = 2 \dot{v} + 2 v^2 - 4(a + c + d) v + k .$$

On eliminating y between (76.2-3) , one finds

$$\begin{aligned}
 \ddot{v} - \frac{2\dot{v}^2}{3v} &= - \left[\frac{2}{3} v + 2(d - a) - \frac{4E}{3} + \frac{k}{3v} \right] \dot{v} \\
 (76.4) \quad &+ \frac{2}{3} v^3 - \left(2c - 2d - \frac{4E}{3} \right) v^2 - \left[\frac{16E^2}{3} + \frac{k}{3} - 4E(c + 2d) - 2\dot{d} \right] v \\
 &+ \frac{8}{3} E k - (c + 2d) k - \frac{\dot{k}}{2} - \frac{k^2}{3v} ,
 \end{aligned}$$

where $E = a + c + d$.

This equation (76.4) is of type III and class E.19 ; to obtain a canonical form , we use the usual method (see § 53) and assume $b_1 + 5 a_1 = 0$ (notations of § 53) , i.e. ,

$$(76.5) \quad c + 3 a = 0 .$$

Then $E = d - 2 a$, $\dot{k} = - 2 a k$ so that (76.4) may be rewritten as

$$\begin{aligned}
 \ddot{v} - \frac{2}{3} \frac{\dot{v}^2}{v} &= - \frac{2}{3} v \dot{v} - \frac{2}{3} (a + d) \dot{v} - \frac{k}{3v} \dot{v} \\
 (76.6) \quad &+ \frac{2}{3} v^3 + \frac{10}{3} (a + d) v^2 - \left[\frac{k}{3} + 4(a + d) - \frac{8}{3} (a + d)^2 \right] v \\
 &+ \frac{2}{3} (a + d) k + \dot{k} - \frac{k^2}{3v} .
 \end{aligned}$$

According to (53.4) , set

$$q = - (a + d) , \quad r = - \frac{k}{3} ;$$

then , equation (76.6) becomes

$$\begin{aligned}
 \ddot{v} - \frac{2}{3} \frac{\dot{v}^2}{v} = & -\frac{2}{3} v \dot{v} + \frac{2}{3} q \dot{v} + r \frac{\dot{v}}{v} + \frac{2}{3} v^3 - \frac{10}{3} q v^2 \\
 (76.7) \quad & + b_2 v + b_3 - \frac{3r^2}{v} ,
 \end{aligned}$$

where

$$b_2 = \frac{8}{3} q^2 + r + 4 \dot{q} , \quad b_3 = 2 q r - 3 \dot{r} .$$

For v (and y) to be stable , one must have (see § 53)

$$\ddot{q} = 2 q^3 + (Kx + H) q + K_1 ,$$

$$3 r = K x + H + 2 \dot{q} + 2 q^2 ;$$

the general solution of (76.7) is

$$(76.8) \quad v = \frac{\dot{w} - \dot{q} + w^2 - q^2}{w - q} ,$$

where w is a solution of

$$\ddot{w} = 2 q^3 + (Kx + H) q + K_1 .$$

77. P2 ; Z2 ; U1 . For P2 , one has $d = 0$. The stable equation for this class is

$$(77.1) \quad \ddot{y} = \frac{3}{4} \left(\frac{1}{y} + \frac{1}{y-1} \right) \dot{y}^2 + \left(2a - \frac{c}{y-1} \right) \dot{y} + y(y-1) \left[\frac{k}{y} + \frac{l}{y-1} - \frac{c^2}{(y-1)^2} \right]$$

where

$$\dot{k} = 2k(2a + c) ,$$

$$\dot{l} = 2 \dot{c} - 2c(2a + c) .$$

The solution of (77.1) is obtained on setting

$$\dot{y} = 2 c y - 2 v(y-1)$$

and is given by (76.7) and (76.8) , where $q = -a$, $r = -\frac{k}{3}$.

XI. Equations of the type $A(x,y) \equiv \frac{2}{3y} + \frac{1}{2(y-1)} .$

78. The stable equations of this type are

$$(78.1) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + P(x,y) \dot{y} + y(y-1) Q(x,y) ,$$

where $P(x,y)$, $Q(x,y)$ are given by (62.2-3) .

The values of y for which the general existence theorem does not apply are 0 , 1 , ∞ .

The solution $y(x)$ of (78.1) may have simple or double parametric poles .

If $y(x)$ has a simple parametric pole , set $y = \frac{s}{z}$,
 $\dot{z} = 1 + u z$, where s is given by $es^2 - as - \frac{5}{6} = 0$ and
 $p = \frac{2}{3} + as$; then $n = -6$ (see 14.4 , case vii) and
 $8e = 3a^2$. Accordingly , $y(x)$ may have two families of simple parametric poles corresponding respectively to $p = 0$, $3as = -2$ and $p = 4$, $3as = 10$. Then , substitution in (78.1) yields

$$zu = \frac{1}{3} \frac{\dot{s}}{s} - \frac{1}{2s} + d - as + (e - h) s + O(z) .$$

The condition for stability for $p = 0$ is thus

$$(78.2) \quad 2h = 3a - \frac{3}{2} a^2 - 3ad .$$

If $y(x)$ has a double parametric pole , then $a = e = 0$

and as a consequence $h = 0$ [see § 17] .

Consider now $y = 0$; the parametric zeros of $y(x)$ may be simple or double .

If $y(x)$ has a simple parametric zero , set $y = sz$,
 $\dot{z} = 1 + u z$ in (78.1) and determine s by

$$\frac{2}{3} + \frac{b}{s} - \frac{f}{s^2} = 0 .$$

Then

$$z\dot{u} = \left(\frac{1}{3} + \frac{b}{s} \right) u - \frac{2}{3} \frac{\dot{s}}{s} - \frac{s}{2} + c + d + \frac{b \dot{s}}{s^2} + \frac{f - k}{s} + O(z) .$$

Because $p = \frac{1}{3} + \frac{b}{s}$ must be an integer , one sees at once

[see § 16 ; note that $n = -3$] that $f = 3b^2$ and that $y(x)$ may have two families of simple parametric zeros corresponding respectively to $p = 0$, $s + 3b = 0$ and $p = 1$, $2s = 3b$.

The condition for stability for $p = 0$ is

$$(78.3) \quad k = 3 \dot{b} - \frac{3}{2} b^2 - 3b(c + d) .$$

If $y(x)$ has a double parametric zero , then $b = f = 0$ and as a consequence $k = 0$.

Finally , consider $y = 1$; the parametric unities of $y(x)$ may be simple or double .

If $y(x)$ has a simple parametric unity , set $y = 1 + sz$,
 $\dot{z} = 1 + z u$ so that (78.1) yields

$$z\dot{u} = \frac{1}{z} \left(\frac{1}{2} - \frac{c}{s} - \frac{g}{s^2} \right) - \frac{c}{s} u + A_0 + O(z) ,$$

where

$$A_0 = -\frac{\dot{s}}{s} + \frac{2}{3}s + a + b + d + \frac{\ell - g}{s}.$$

Now, determine s by $\frac{1}{2} - \frac{c}{s} - \frac{g}{s^2} = 0$; because $p = -\frac{c}{s}$ must be an integer, one has $c = 0$, $g \neq 0$, $s^2 = 2g$ and $p = 0, 0$. The condition of stability is then $A_0 = 0$ and according to $s^2 = 2g$,

$$\frac{2}{3}s^2 + \ell - g = 0, \quad \frac{\dot{s}}{s} = a + b + d$$

or

$$(78.4) \quad 3\ell + g = 0,$$

$$(78.5) \quad \dot{g} = 2g(a + b + d).$$

If $y(x)$ has a double parametric unity, then $c = g = 0$ and consequently $\ell = 0$.

Note that c is always zero. Because of the asymmetric role of $y = 0, 1, \infty$, one has to consider eight distinct possibilities, namely

$$(P2; Z2; U2 \text{ or } U1) \quad ; \quad (P2; Z1; U2 \text{ or } U1) \quad ;$$

$$(P1; Z2; U2 \text{ or } U1) \quad ; \quad (P1; Z1; U2 \text{ or } U1) \quad .$$

The conditions for stability are given above except for $P1$, $p = 4$ and $Z1$, $p = 1$.

We begin by considering the case $(P1 ; Z1 ; U1)$.

79. $P1 ; Z1 ; U1$. On using the conditions for stability already obtained in paragraph 78 and, for convenience, changing ℓ into 3ℓ , one may rewrite equation 78.1 as

$$\begin{aligned}
 \ddot{y} = & \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + \left(ay + \frac{b}{y} + d \right) \dot{y} + \\
 (79.1) \quad & + 3y(y-1) \left[\frac{a^2}{8} y + \frac{b^2}{y^2} + \frac{3\ell}{(y-1)^2} \right] + \\
 & + y(y-1) \left[h + \frac{k}{y} + \frac{3\ell}{y-1} \right] ,
 \end{aligned}$$

where h , k and g are given by 78.2-3-5 respectively and where

$$(79.2) \quad \ell + g = 0 .$$

Now , we reduce the problem to another one involving an equation with only one singular value , namely $y = \infty$.

To do this , note that $\frac{\dot{y}}{y} - \frac{3a}{2} y$ is regular for $P_1, p = 0$

and that $\frac{\dot{y}}{y} + \frac{3b}{y}$ is regular for Z_1 , $p = 0$; accordingly , set

$$(79.3) \quad \dot{y} = \frac{3a}{2} y(y-1) + 3b(y-1) + 3vy$$

and substitute in (79.1) . For convenience , set first

$$(79.4) \quad \dot{y} = 3b(y-1) + 3w_1 y$$

and find

$$\begin{aligned}
 \dot{w}_1 = & \left[1 + \frac{3}{y-1} \right] \frac{w_1^2}{2} + (b+d+ay) w_1 + ab(y-1) \\
 (79.5) \quad & + (y-1) \left[\frac{a^2}{8} y + \frac{3\ell}{(y-1)^2} + \frac{\dot{a}}{2} - \frac{a^2}{4} - \frac{ad}{2} + \frac{\ell}{y-1} \right] ;
 \end{aligned}$$

then , write

$$(79.6) \quad w_1 = \frac{a}{2} (y-1) + v$$

so that

$$(79.7) \quad \dot{v} - \frac{v^2}{2} - (a+b+d)v - \ell = \frac{3}{y-1} \left[\frac{v^2}{2} + \ell \right] .$$

Now set

$$(79.8) \quad \begin{cases} a + b + d = -r , \\ A = 2 \dot{v} - v^2 + 2r v - 2\ell , \\ B = v^2 + 2\ell ; \end{cases}$$

then , according to (78.4-5-7) , one obtains

$$(79.9) \quad \dot{\ell} = -2\ell r ,$$

$$(79.10) \quad y - 1 = \frac{3B}{A} .$$

To eliminate y between (79.3) and (79.10) , rewrite (79.3) as

$$\frac{\dot{y}}{y-1} = 3 \frac{a}{2} + 3b + 3v + \frac{3a}{2}(y-1) + \frac{3v}{y-1}$$

and note that

$$(79.11) \quad \dot{B} - v A = (v - 2r)B ;$$

then v satisfies

$$(79.12) \quad \ddot{v} = -v\dot{v} + v^3 - \frac{3}{2}(a+2b+2r)\dot{v} - \left[\frac{3}{2}(a-b)+r \right] v^2 - \left[\dot{r} + \frac{3}{2}(a+2b)r - 2\ell + 2r^2 \right] v - 3(a-b)\ell .$$

To integrate this equation of type I and class 20.6 , set $v = w - q$ and determine q so that the coefficient of \dot{w} be three times the coefficient of w^2 ; one finds

$$(79.13) \quad q = -\frac{3}{2} \left(\frac{a}{5} - \frac{b}{2} \right) ,$$

$$(79.14) \quad \ddot{w} = -w\dot{w} + w^3 + \left(\frac{2a}{5} + \frac{b}{4} + d \right) (3\dot{w} + w^2) + C_1 w + C_2 ,$$

where C_1 , C_2 are independent of w .

By the method already used for the equations of type $\frac{2}{3} \left(\frac{1}{y} + \frac{1}{y-1} \right)$ [i.e. a proper choice of φ ; see § 64] ,

one may assume

$$(79.15) \quad \frac{2a}{5} + \frac{b}{4} + d = 0$$

so that

$$(79.16) \quad \begin{cases} 9a = -10(q+r) , & 9b = 4(2q-r) , \\ C_1 = \dot{q} - 3q^2 - (\dot{r} + qr - r^2 - 2\ell) , \\ C_2 = \ddot{q} + 2q^3 + (\dot{r} + rq - r^2 - 2\ell)q + 6\ell q + 2r\ell . \end{cases}$$

If equation (79.14) is to be stable [see § 25.c] , one may have

$$(79.17) \quad C_1 = -12 V_1 , \quad C_2 = 12 \dot{V}_1 ,$$

where V_1 is a solution of

$$(79.18) \quad \ddot{V} = 6 V^2 + K x + H ;$$

equation (79.14) is then

$$(79.19) \quad \ddot{w} = -w\dot{w} + w^3 - 12 V_1 w + 12 \dot{V}_1 .$$

Now set

$$(79.20) \quad 2\ell = -s^2$$

and observe that , because of (79.9) ,

$$(79.21) \quad \dot{s} + s r = 0 , \quad \ddot{s} + s(\dot{r} - r^2) = 0 ;$$

one has

$$s(\dot{r} + qr - r^2 - 2\ell) = -\ddot{s} - q\dot{s} + s^3 ;$$

then , from (79.16-17) , one deduces

$$\ddot{s} = -s\dot{q} - \dot{s}q - 12V_1s + s^3 + 3sq^2 ,$$

$$\ddot{q} = -q\dot{q} - s\dot{s} - 12V_1q + q^3 + 3s^2q + 12\dot{V}_1$$

so that $q + s$ and $q - s$ are solutions of (79.19) . According to § 25.c , one has

$$q + s = \frac{\dot{V}_3 - \dot{V}_1}{V_3 - V_1} , \quad q - s = \frac{\dot{V}_2 - \dot{V}_1}{V_2 - V_1} ,$$

where V_1 , V_2 , V_3 are distinct solutions of (79.18) .

The solution of (79.1) is given by

$$y - 1 = \frac{3(v^2 - s^2)}{2\dot{v} - v^2 + 2rv + s^2} ,$$

where $r = -\frac{\dot{s}}{s}$, $v = w - q$ and where

$$w = \frac{\dot{V} - \dot{V}_1}{V - V_1}$$

is a solution of (79.19) .

80. We now consider the other cases .

1. P2 ; Z2 ; U2 . The equation is

$$(80.1) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + d\dot{y} .$$

A transformation $y \rightarrow \varphi(x)$ may be chosen so that $d = 0$

[set $\ddot{\varphi} = d\dot{\varphi}$] ; then , the integral of (80.1) is given by

$$(80.2) \quad \dot{y}^6 = K y^4 (y - 1)^3 ,$$

where K is as usual , an arbitrary constant . Therefore , $y(x)$ is an elliptic function of x and is stable .

ii. P2 ; Z2 ; U1 . The equation is

$$(80.3) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + d \dot{y} - \frac{gy}{y-1} + \ell y ;$$

by a proper choice of the transformation $x \rightarrow \varphi(x)$, one may suppose $d = 0$; hence , [see 78.4-5] $\dot{g} = 0$, $g = -3\ell$, ℓ a constant .

To integrate (80.3) , where $d = 0$, set $y = w^{-1}$; then

$$\ddot{w} = \left[\frac{5}{6w} + \frac{1}{2(w-1)} \right] \dot{w}^2 + 3\ell \frac{w^2}{w-1} - \ell w$$

or on setting $w = z^3$,

$$\ddot{z} = \left[\frac{1}{2z} + \frac{3}{2} \frac{z^2}{(z^3-1)} \right] \dot{z}^2 + \frac{\ell z^4}{z^3-1} - \frac{\ell}{3} z .$$

The integral of this equation may be obtained by the method of variation of parameters . To this end , set

$$\dot{z}^2 = z(z^3 - 1) K(z)$$

and determine $K(z)$ by

$$\frac{dK}{dz} = - \frac{2\ell}{3} \frac{d}{dz} \left(\frac{z}{z^3-1} \right)$$

so that

$$\dot{z}^2 = K_1 z (z^3 - 1) - \frac{2\ell}{3} z^2$$

where K_1 is an arbitrary constant . Therefore , $z(x)$ is an

elliptic function and $w(x)$ is stable .

81. P2 ; Z1 ; U2 . The equation is (78.1) , where $a = h = \ell = 0$, i.e. ,

$$(81.1) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + \left(\frac{b}{y} + d \right) \dot{y} + (y-1) \left[\frac{3b^2}{y} + k \right] ,$$

where k is given by (78.3) .

We use the notations of paragraph 78. According to (79.4) , set

$$\dot{y} = 3b(y-1) + 3vy$$

so that [see (79.10)]

$$(81.2) \quad y - 1 = \frac{3 v^2}{2v - v^2 + 2(b+d)v} .$$

By a proper choice of $x \rightarrow \varphi(x)$, one may suppose $b + 4d = 0$; then ,

$$(81.3) \quad 4q = 3b , \quad r + q = 0 ,$$

$$(81.4) \quad v = w - q ,$$

$$(81.5) \quad \ddot{w} = -w\dot{w} + w^3 + C_1 w + C_2 ,$$

$$(81.6) \quad C_1 = 2\dot{q} - q^2 = -12V_1 ,$$

$$(81.7) \quad C_2 = \ddot{q} - q\dot{q} = 12\dot{V}_1 .$$

Differentiation of (81.6) and comparison with (81.7) show that V_1 is a constant ; therefore , equation (79.18) reduces to

$$(81.8) \quad \ddot{V} = 6V^2 + H .$$

On setting $\dot{q} = -2 \frac{\dot{Q}}{Q}$, one sees from (81.6) that Q is determined by

$$(81.9) \quad \ddot{Q} = 3 V_1 Q .$$

The integral of (81.1) is given by (81.2) and (81.4), where

$$(81.10) \quad w = \frac{\dot{V}}{V - V_1} ;$$

in (81.10), V , the solution of (81.8), is an elliptic function.

82. P2 ; Z1 ; U1. The equation is (79.1) with $a=h=0$, i.e.,

$$(82.1) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + \left(\frac{b}{y} + d \right) \dot{y} + 3y(y-1) \left[\frac{b^2}{y^2} + \frac{3\ell}{(y-1)^2} \right] \\ + k(y-1) + 3\ell y .$$

Using again the method of paragraph 79, one finds

$$q + r = 0 ,$$

$$(82.2) \quad C_1 = 2 \dot{q} - q^2 + 2\ell = -12 V_1 ,$$

$$(82.3) \quad C_2 = \ddot{q} - q \dot{q} + 2\ell q = 12 \dot{V}_1$$

so that

$$3 \ddot{q} - 3 q \dot{q} + 2 \dot{\ell} + 2 \ell q = 0 .$$

Because $\dot{\ell} = 2 \ell q$ [see (79.9)], one also has

$$(82.4) \quad \ddot{q} - q \dot{q} + 2 \ell q = 0$$

so that $C_2 = 0$ and V_1 is a constant ; therefore, equation (79.18) reduces to

$$(82.5) \quad \ddot{V} = 6 V^2 + H .$$

Now it follows from (82.2) and (82.4) that

$$(82.6) \quad \ddot{q} - 3 q \dot{q} + q^3 - 12 V_1 q = 0 .$$

On setting $q = -\frac{\dot{Q}}{Q}$, equation (82.6) becomes the linear

equation of the third order

$$(82.7) \quad Q = 12 V_1 \dot{Q}$$

and ℓ is given by

$$(82.8) \quad \ell Q^2 = K_1 ,$$

where K_1 is an arbitrary constant .

The integral of (82.1) is given by

$$y - 1 = 3 \frac{v^2 + 2 \ell}{2v - v^2 - 2qv - 2 \ell} ,$$

$$v = w - q , \quad w = \frac{\dot{V}}{V - V_1} ,$$

where V , the solution of (82.5) , is an elliptic function .

83. P1 ; Z2 ; U2 . The equation is (78.1) with $b=f=k=g=\ell=0$,
i.e. ,

$$(83.1) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + (ay+d)\dot{y} + 3 \frac{a^2}{8} y^2 (y-1) \\ + \frac{3y}{2} (y-1) \left[\dot{a} - \frac{a^2}{2} - ad \right] .$$

We again use transformation (79.3) and find

$$(83.2) \quad y - 1 = \frac{3 v^2}{v + 2rv - v^2}$$

with $r = - (a + d)$. Now set

$$(83.3) \quad v = w - q \quad , \quad 2 q = r$$

so that

$$(83.4) \quad \ddot{w} = - w \dot{w} + w^3 + C_1 w + C_2 \quad ,$$

where

$$C_1 = - \dot{q} - q^2 = - 12 V_1 \quad ,$$

$$C_2 = \ddot{q} + 2 q \dot{q} = 12 \dot{V}_1 \quad ;$$

V_1 is a solution of (79.18) .

Therefore , the integral of (83.1) is given by (83.2-3-4) ,

where $q = \frac{\dot{Q}}{Q}$ is given by

$$\ddot{Q} = 12 V_1 Q \quad .$$

84. P1 ; Z2 ; U1 . The stable equation for this class is

(79.1) with $b = f = k = 0$, i.e. ,

$$(84.1) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + (ay+d)\dot{y} + 3y(y-1) \left[\frac{a^2}{8} y + \frac{3\ell}{(y-1)^2} \right] .$$

We use again (79.3) and find

$$(84.2) \quad y - 1 = \frac{3(v^2 + 2\ell)}{2v - v^2 + 2vr - 2\ell}$$

with

$$(84.3) \quad r = - (a + d) \quad , \quad \dot{\ell} = - 2\ell r \quad .$$

Set

$$(84.4) \quad v = w - q, \quad 2q = r$$

so that

$$(84.5) \quad \ddot{w} = -w \dot{w} + w^3 + C_1 w + C_2,$$

where

$$(84.6) \quad C_1 = -\dot{q} - q^2 + 2\ell = -12 \dot{v}_1,$$

$$(84.7) \quad C_2 = \ddot{q} + 2q \dot{q} + 8\ell q = 12 \ddot{v}_1;$$

v_1 is a solution of (79.18).

On eliminating ℓ between (84.6) and (84.7), one obtains

$$\ddot{q} + 6q \dot{q} + 4q^3 - 48 \dot{v}_1 q - 12 \ddot{v}_1 = 0.$$

Or, on setting $2q = Q$,

$$\ddot{Q} = -3Q \dot{Q} - Q^3 + 48 \dot{v}_1 Q + 24 \ddot{v}_1.$$

This equation is of type I and class 20.4; by the transformation $Q = \frac{\dot{U}}{U}$, it reduces to

$$U - 48 \dot{v}_1 \dot{U} - 24 \ddot{v}_1 U = 0.$$

From (84.3), it follows that $\ell U^2 = K_1$, where K_1 is an arbitrary constant.

The integral of (84.1) is given by (84.2-4) with $q = \frac{\dot{U}}{2U}$.

85. P1; Z1; U2. The stable equation for this class is

(79.1) with $\ell = 0$, i.e.,

$$(85.1) \quad \ddot{y} = \left[\frac{2}{3y} + \frac{1}{2(y-1)} \right] \dot{y}^2 + \left(ay + \frac{b}{y} + d \right) \dot{y} + y(y-1) \left[\frac{3a^2}{8} y + \frac{3b^2}{y^2} + h + \frac{k}{y} \right]$$

We again use the same method . One has

$$(85.2) \quad y - 1 = \frac{3 v^2}{2v - v^2 + 2rv} .$$

On setting

$$(85.3) \quad v = w - q ,$$

one finds

$$(85.4) \quad \ddot{w} = - w \dot{w} + w^3 + C_1 w + C_2 ,$$

where

$$(85.5) \quad C_1 = \dot{q} - 3 q^2 - (\dot{r} + q r - r^2) = - 12 V_1 ,$$

$$(85.6) \quad C_2 = \ddot{q} + 2 q^3 + (\dot{r} + r q - r^2) q = 12 \dot{V}_1 ;$$

V_1 is a solution of (79.18) .

On eliminating $\dot{r} + q r - r^2$ from (85.5-6) , one sees that q is a solution of (85.4) . Further , on setting

$$q - r = \frac{\dot{U}}{U}$$

in (85.5) , one obtains

$$\ddot{U} + q \dot{U} - (3 q^2 - 12 V_1) U = 0$$

which determines r when q is given .

The integral of (85.1) is determined by (85.2-3) .

XII. Equations of the type $A(x,y) = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-H} \right)$

86. The stable equations of this type are

$$(86.1) \quad \ddot{y} = \frac{\dot{y}^2}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-H} \right) + \frac{M(x,y)}{y(y-1)(y-H)} + \frac{N(x,y)}{y(y-1)(y-H)},$$

where $M(x,y)$, $N(x,y)$ are polynomials in y of degrees 4 and 6 respectively; H is a constant distinct from 0 and 1 or $H = x$.

For convenience, we rewrite equation (86.1) as

$$(86.2) \quad \ddot{y} = \frac{\dot{y}^2}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-H} \right) + \left[ay + a_1 + \frac{b}{y} - \frac{c}{y-1} + \frac{d}{y-H} \right] \dot{y} \\ + y(y-1)(y-H) \left[\frac{e}{y^2} - \frac{f}{(y-1)^2} + \frac{g}{(y-H)^2} + h + \frac{k}{y} + \frac{l}{y-1} + \frac{m}{y-H} \right],$$

where a , a_1 , ..., m are analytic functions of x .

The values of y for which the general existence theorem does not apply are $y = \infty$, 0 , 1 , H .

If a and h are not both zero, $y(x)$ has simple parametric poles; if $a = h = 0$, $y(x)$ has double parametric poles.

If $y(x)$ has a simple parametric pole, set $y = \frac{s}{z}$,

$\dot{z} = 1 + uz$. According to (14.4), one has $n = -2$ and hence

[see (16.22)]

$$(86.3) \quad a = 0, \quad h \neq 0, \quad p = 0, \quad 2hs^2 = 1.$$

The corresponding condition for stability follows from

$$z \dot{u} = \frac{\dot{s}}{s} + a_1 - (k + \ell + m) s + O(z)$$

and is

$$\frac{\dot{s}}{s} + a_1 - (k + \ell + m) s = 0$$

or , because $2 h s^2 = 1$,

$$(86.4) \quad k + \ell + m = 0$$

and

$$(86.5) \quad \frac{\dot{s}}{s} + a_1 = 0 \quad \text{or} \quad \dot{h} = 2 h a_1 .$$

If $y(x)$ has a double parametric pole , one has $a = h = 0$ and thus $k + \ell + m = 0$ (see § 17) .

The values $y = 0$, $y = 1$ play the same role ; the transformation $y = 1 - w$ brings equation (86.2) to an equation of the same form according to the equivalence table

(y) :	H	a	a_1	b	c	d	e	f	g	h	k	ℓ	m
(w) :	$1-H$	$-a$	a_1+a	c	b	$-d$	$-f$	$-e$	g	h	$-\ell$	$-k$	$-m$

If $y(x)$ has a simple parametric zero , set $y = sz$, $\dot{z} = 1 + uz$; then (86.2) gives

$$z \dot{u} = \frac{1}{z} \left(\frac{1}{2} + \frac{b}{s} + \frac{eH}{s^2} \right) + \frac{b}{s} u + A_0 + O(z) ,$$

where

$$A_0 = - \frac{\dot{s}}{s} - \left(1 + \frac{1}{H} \right) \frac{s}{2} + a_1 + c - \frac{d}{H} + b \frac{\dot{s}}{s^2} - \frac{e}{s} (1+H) + \frac{k}{s}$$

[according to (86.3) , one has $a = 0$] .

Now determine s by

$$\frac{1}{2} + \frac{b}{s} + \frac{eH}{s^2} = 0 ;$$

because $p = \frac{b}{s}$ must be an integer, one concludes that

$$\begin{aligned} b = 0, \quad e \neq 0, \quad p = 0, \quad s^2 + 2eH = 0; \\ \text{the condition for stability is } A_0 = 0 \text{ or because } s^2 = -2eH, \\ (86.6) \quad -\frac{\dot{s}}{s} + a_1 + c - \frac{d}{H} + k \frac{H}{s} = 0. \end{aligned}$$

Because $s^2 = -2eH$, one has

$$(86.7) \quad k = 0,$$

$$(86.8) \quad 2(a_1 + c - \frac{d}{H}) = \frac{\dot{e}}{e} + \frac{\dot{H}}{H}.$$

If $y(x)$ has a double parametric zero, then $b = e = k = 0$,
[see § 17].

According to the table of equivalence, one obtains the conditions for stability

$$\begin{aligned} \text{for U1: } c = 0, \quad f \neq 0, \\ (86.9) \quad \ell = 0, \end{aligned}$$

$$(86.10) \quad 2 \left(a_1 + b + \frac{d}{1-H} \right) = \frac{\dot{f}}{f} + \frac{\dot{H}}{1-H};$$

$$\text{for U2: } c = f = \ell = 0.$$

87. We have to consider two cases according as to whether H is a constant or $H = x$.

Suppose H is a constant distinct from 0 and 1. The transformation $y = H(1 - v)$ brings equation (86.1) to an equation of the same form according to the table of equivalence

$$\begin{array}{l}
 (y) : \quad H \quad a \quad a_1 \quad b \quad c \quad d \quad e \quad f \quad g \quad h \quad k \quad \ell \quad m \\
 (v) : 1 - \frac{1}{H} \quad -aH \quad aH+a_1 \quad -\frac{d}{H} \quad \frac{b}{H} \quad \frac{c}{H} \quad g \quad -e \quad -f \quad hH^2 \quad -mH \quad -kH \quad -\ell m
 \end{array}$$

[Note that $y = 0, 1, H$ correspond to $v = 1, 1 - \frac{1}{H}, 0$ respectively] .

Therefore , $y = H$ plays the same role as $y = 0$ or $y = 1$.

The conditions for stability for H follow readily from the preceding paragraph ; one obtains

$$\begin{array}{l}
 \text{for H1} : \quad d = 0 \quad , \quad g \neq 0 \quad , \\
 (87.1) \quad \quad \quad m = 0 \quad ,
 \end{array}$$

$$(87.2) \quad 2 \left(a_1 - \frac{c}{H-1} \right) = \frac{\dot{g}}{g} \quad ;$$

$$\text{for H2} : \quad d = g = m = 0 .$$

Moreover , a transformation $x \rightarrow \varphi(x)$ may be chosen so that $a_1 = 0$ [set $\ddot{\varphi} = a_1 \dot{\varphi}$] . It then follows that c, f, g, h are constant or eventually zero [see (86.5-8-10) ; (87.2)] .

The stable equation of the type considered is then

$$(87.3) \quad \ddot{y} = \frac{\dot{y}^2}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-H} \right) + y(y-1)(y-H) \left[\frac{e}{y^2} - \frac{f}{(y-1)^2} + \frac{g}{(y-H)^2} + h \right] ,$$

where e, f, g, h are constant .

To integrate this equation , we use the method of variation of parameters . One obtains

$$\dot{y}^2 = 2y(y-1)(y-H) \left[K_1 + h y - \frac{e}{y} + \frac{f}{y-1} - \frac{g}{y-H} \right] ,$$

where K_1 is an arbitrary constant ; this equation shows that

$y(x)$ is an elliptic function and is stable .

The result holds if one or more of the constants e , f , g , h is zero .

88. Suppose $H = x$. Taking into account the conditions for stability already obtained for $y = \infty$, 0 , 1 [i.e. $a = b = c = 0$, $k = \ell = m = 0$] , one may rewrite equation (86.1) as

$$\ddot{y} = \frac{\dot{y}^2}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) + \left(a_1 + \frac{d}{y-x} \right) \dot{y} \quad (88.1)$$

$$+ y(y-1)(y-x) \left[\frac{e}{y^2} - \frac{f}{(y-1)^2} + \frac{g}{(y-x)^2} + h \right] .$$

Set $y = x + s z$, $\dot{z} = 1 + z u$; equation (88.1) yields

$$z \dot{u} = \frac{1}{2 s^2 z} \left[(1+s)^2 + 2d(1+s) + 2x(x-1)g \right] + \frac{1+d}{s} u + A_x + O(z) ,$$

where

$$A_x = - 2 \frac{\dot{s}}{s} + \left(a_1 \frac{1+s}{s} + d \frac{\dot{s}}{s^2} \right) + (2x-1) \frac{g}{s} + \quad (88.2)$$

$$+ \frac{1+s}{2s} \left[\frac{2\dot{s}}{s} + (1+s) \left(\frac{1}{x} + \frac{1}{x-1} \right) \right] .$$

Determine s by

$$(88.3) \quad (1+s)^2 + 2d(1+s) + 2x(x-1)g = 0$$

and note that

$$(88.4) \quad p = \frac{1+d}{s}$$

must be an integer .

Elimination of s between (88.3-4) gives

$$(88.5) \quad \left(1 + \frac{d+1}{p}\right)^2 + 2d \left(1 + \frac{d+1}{p}\right) + 2x(x-1)g = 0 .$$

Taking into account the value of g given by (88.5) , one rewrites (88.3) as

$$\left(s - \frac{d+1}{p}\right) \left(s + 2 + \frac{d+1}{p} + 2d\right) = 0$$

so that

$$\frac{1+d}{s} = \begin{cases} p \\ -\frac{p}{2p+1} = q . \end{cases}$$

The integers p , q satisfy the Diophantine equation

$$p + q + 2pq = 0$$

so that $p = q = 0$.

Accordingly ,

$$(88.6) \quad d = -1 ,$$

$$(88.7) \quad s^2 = 1 - 2x(x-1)g .$$

The corresponding condition for stability is $A_x = 0$ or

$$(88.8) \quad 2a_1 + 2(2x-1)g + (1+s^2) \left(\frac{1}{x} + \frac{1}{x-1}\right) = 0 ,$$

$$(88.9) \quad -\frac{s}{s} + a_1 + \frac{1}{x} + \frac{1}{x-1} = 0 .$$

From (88.7-8) , and (88.9) , one obtains

$$(88.10) \quad a_1 + \frac{1}{x} + \frac{1}{x-1} = 0 ,$$

$$\dot{s} = 0 .$$

Therefore , $1 - 2x(x-1)g$ or $x(x-1)g$ is a constant and

$$(88.11) \quad g = \frac{1 - K}{2x(x-1)} ,$$

where K is an arbitrary constant .

The values of h , e , f are now given by (86.5-8-10) and are

$$(88.12) \quad h = \frac{k_1}{x^2(x-1)^2} , \quad e = \frac{k_2}{x(x-1)^2} , \quad f = \frac{k_3}{x^2(x-1)} ,$$

where k_1 , k_2 , k_3 are arbitrary constants .

The equation is then

$$(88.13) \quad \ddot{y} = \frac{\dot{y}^2}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) - \left(\frac{1}{y-x} + \frac{1}{x} + \frac{1}{x-1} \right) \dot{y} \\ + \frac{y(y-1)(y-x)}{2x^2(x-1)^2} \left[k_1 + \frac{k_2 x}{y^2} + \frac{k_3(x-1)}{(y-1)^2} + \frac{(1-k_4)x(x-1)}{(y-x)^2} \right] ;$$

k_1 , k_2 , k_3 , k_4 are again arbitrary constants .

That the equation (88.13) is stable has been shown by Painlevé . Equation (88.13) is the irreducible equation VI of Table I .

Bibliography .

1. F.J. Bureau :
 - a) Sur la recherche des équations différentielles du second ordre dont l'intégrale générale est à points critiques fixes .
(Bulletin de la Classe des Sciences de l'Académie royale de Belgique 5^e s. t XXV , 1939, pp. 51-68).
2. H.T. Davis and others :
 - a) Studies in differential equations .
(Northwestern University Press 1956).
3. B. Gambier :
 - a) Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes .
(Thèse Paris , 1909 ; Acta Mathem. t. 39 , pp. 1-55).
4. P. Painlevé :
 - a) Notice sur les travaux scientifiques .
(Paris 1900) .
 - b) Leçons sur la théorie analytique des équations différentielles .
(Paris 1897) .
 - c) Sur les équations différentielles dont l'intégrale générale est uniforme .
(Bulletin de la Société Mathématique de France , t. 28 , 1900 pp. 201-261).
 - d) Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme .
(Acta Mathematica , t. 25 , sept. 1900 pp. 1-80).